

ON THE DISCRETISATION IN TIME OF THE STOCHASTIC ALLEN–CAHN EQUATION

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ABSTRACT. We consider the stochastic Allen–Cahn equation perturbed by smooth additive Gaussian noise in a spatial domain with smooth boundary in dimension $d \leq 3$, and study the semidiscretisation in time of the equation by an Euler type split-step method. We show that the method converges strongly with a rate $O(\Delta t^{\frac{1}{2}})$. By means of a perturbation argument, we also establish the strong convergence of the standard backward Euler scheme with the same rate.

1. INTRODUCTION

Let $\mathcal{D} \subset \mathbb{R}^d$, $d \leq 3$, be a spatial domain with smooth boundary $\partial\mathcal{D}$ and consider the stochastic partial differential equation written in the abstract Itô form

$$(1.1) \quad dX + AX \, dt + f(X) \, dt = dW, \quad t \in (0, T]; \quad X(0) = X_0,$$

where $\{W(t)\}_{t \geq 0}$ is an $L_2(\mathcal{D})$ -valued Q -Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ with respect to the normal filtration $\{\mathcal{F}_t\}_{t \geq 0}$. We use the notation $H = L_2(\mathcal{D})$ with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$ and $V = H_0^1(\mathcal{D})$ with norm denoted by $\|\cdot\|_1$. Moreover, $A: V \rightarrow V'$ denotes the linear elliptic operator $Au = -\nabla \cdot (\kappa \nabla u)$ for $u \in V$, where $\kappa(\xi) \geq \kappa_0 > 0$ is smooth. As usual we consider the bilinear form $a: V \times V \rightarrow \mathbb{R}$ defined by $a(u, v) = (Au, v)$ for $u, v \in V$, and (\cdot, \cdot) denotes the duality pairing of V' and V . We denote by $\{E(t)\}_{t \geq 0}$ the analytic semigroup in H generated by the realisation of $-A$ in H with $D(A) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$. Finally, $f: D_f \subset H \rightarrow H$ is given by $f(u)(x) = F'(u(x))$, where $F(s) = \frac{1}{4}(s^2 - \beta^2)^2$ is a double well potential. Note that f is only locally Lipschitz and does not satisfy a linear growth condition. It does, however, satisfy a global one-sided Lipschitz condition (1.6) and a local Lipschitz condition (1.7), see below.

We consider a fully implicit split-step scheme for the temporal discretisation of (1.1) via the iteration, for $j = 0, 1, \dots, N-1$,

$$(1.2) \quad X^0 = X_0,$$

$$(1.3) \quad Y^j + \frac{k}{2}AY^j = X^j,$$

$$(1.4) \quad Z^j + kf(Z^j) = Y^j,$$

$$(1.5) \quad X^{j+1} + \frac{k}{2}AX^{j+1} = Z^j + \Delta W^j,$$

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where $k = T/N$ is a step size with $2\beta^2 k < 1$, $t_j = jk$, and $\Delta W^j = W(t_{j+1}) - W(t_j)$. This scheme is implicit also in the drift term f , but, to compute X^{j+1} one only needs to evaluate f at the \mathcal{F}_{t_j} -measurable function Z^j . This is a key point, since it allows the construction of an Itô-process and the use of Itô's formula in the analysis.

The scheme in (1.2)–(1.5) is an attempt to generalise the method of Higham, Mao, and Stuart [17] for SDEs to the infinite-dimensional case. The specific form of the non-linearity, $f(u) = u^3 - \beta^2 u$, considered here, is only used when proving the existence of solutions of (1.4) in Lemma 3.3. Apart from that, the properties of f that will be utilised are the following one-sided Lipschitz condition, local Lipschitz condition, and polynomial growth condition:

$$(1.6) \quad \langle f(x) - f(y), x - y \rangle \geq -\beta^2 \|x - y\|^2,$$

$$(1.7) \quad \|A^{-1/2}(f(x) - f(y))\| \leq P^1(\|x\|_1, \|y\|_1) \|x - y\|,$$

$$(1.8) \quad \|f(x)\| \leq P^2(\|x\|_1),$$

where P^1 and P^2 are polynomials. In our specific case,

$$P^1(s, t) = C_1(t^2 + s^2 + 1), \quad P^2(s) = C_2(t^3 + 1),$$

for some sufficiently large numbers C_1 and C_2 . We also use the fact that $f(0) = 0$ and the dissipativity properties

$$(1.9) \quad \langle f(x), x \rangle \geq -\beta^2 \|x\|^2,$$

$$(1.10) \quad \langle f(x), x \rangle_1 \geq -\beta^2 \|x\|_1^2.$$

The motivation for the three-step scheme is that it has a symmetry property with respect to f that allows us to view it as the Euler–Maruyama scheme for a perturbed equation, see (3.5), where A is replaced by a bounded, positive definite, and self-adjoint operator \mathcal{A}_k , and f is replaced by an operator F_k that is globally Lipschitz continuous and retains the properties (1.6) and (1.9)–(1.10). The smoothing of the operators allows us to use Itô's formula, paving the way for a situation where the one-sided Lipschitz condition helps us to prove moment bounds and establish the correct convergence rate. In contrast to, for example, [20], we thus avoid the use of the mild form of (1.1), which only allows for a proof of the mere fact of strong convergence without rate. How the symmetry comes into play will be clear from the proof of Item (3) of Lemma 3.3.

Given that the solution of (1.2)–(1.5) converges to the solution of (1.1) it is not difficult to prove that the solution of the standard fully implicit backward Euler (BE) scheme for (1.1),

$$(1.11) \quad X_{\text{be}}^j - X_{\text{be}}^{j-1} + kAX_{\text{be}}^j + kf(X_{\text{be}}^j) = \Delta W^{j-1}, \quad j = 1, 2, \dots, N; \quad X_{\text{be}}^0 = X_0.$$

also converges, and with the same rate. The main findings of this paper are summarised in the following theorem. We use the notation $\|\cdot\|_s = \|A^{s/2} \cdot\|$, $s = 1, 2$.

Theorem 1.1. *Let $\{X(t)\}_{0 \leq t \leq T}$, $\{X^n\}_{0 \leq n \leq N}$, and $\{X_{\text{be}}^n\}_{0 \leq n \leq N}$ be the solutions of (1.1), (1.2)–(1.5), and (1.11), respectively. Assume further that $\mathbf{E}\|X_0\|_1^{18} < \infty$, $\mathbf{E}\|X_0\|_2^2 < \infty$, and $\|A^{1/2}Q^{1/2}\|_{\text{HS}} < \infty$. Then, for any $k \leq k_0$ with $2k_0\beta^2 < 1$, there exist $c, C > 0$, depending on T , $\mathbf{E}\|X_0\|_1^{18}$, $\mathbf{E}\|X_0\|_2^2$, $\|A^{1/2}Q^{1/2}\|_{\text{HS}}$, and k_0 ,*

such that

$$(1.12) \quad \mathbf{E} \sup_{0 \leq n \leq N} \|X^n - X(t_n)\|^2 \leq ck,$$

$$(1.13) \quad \mathbf{E} \sup_{0 \leq n \leq N} \|X_{\text{be}}^n - X(t_n)\|^2 \leq Ck.$$

Proof. The first result, (1.12), is a consequence of Theorem 5.3, Theorem 5.4, and the triangle inequality. The second, (1.13), follows from (1.12) and from Theorem 5.8 by using, again, the triangle inequality. \square

Strong convergence results for time discretisation schemes for SPDEs with globally Lipschitz coefficients, or at least some sort of linear growth condition as in [13], are abundant, see, for example, [4, 10, 11, 12, 15, 16] and the references therein. In contrast, there are only few results on strong convergence of time discretisation schemes for SPDEs with superlinearly growing coefficients. Furthermore, almost all of these results are establishing the fact of strong convergence with no rate given [3, 12, 14, 20, 23]. Only in a very recent paper [18] do the authors obtain strong rates, by employing an exponential integrator scheme for a class of SPDEs without a linear growth condition on the coefficients. The methods used in the present paper and the split-step scheme (1.2)–(1.5) itself are completely different from the methods and the numerical scheme in [18] and also give a result for the strong rate of convergence of the classical fully implicit backward Euler method. Finally, we also mention the thesis [19] where, in Chapter 5, weak rate of convergence is obtained for an SPDE with space-time white noise in one spatial dimension and with a non-globally Lipschitz (polynomial) semilinear term.

The paper is organised as follows. In Section 2 we gather some necessary background material on deterministic and stochastic evolution equations. In Section 3 we rewrite the original split-step scheme (1.2)–(1.5) as an explicit Euler discretisation of an auxiliary stochastic differential equation (3.5) involving bounded operators and globally Lipschitz non-linearities. In Lemmata 3.1–3.3 we establish some key properties of the coefficients appearing in (3.5). In Section 4 we first state our hypothesis on the smoothness of the initial data and the regularity of the covariance operator of W . In Section 4.1 we prove moment bounds on the solution of the auxiliary equation (3.5) making fully use of the boundedness of the coefficients so that an Itô formula can be rigorously employed. In Subsection 4.2 we establish various moment bounds on the solution of the split-step scheme (1.2)–(1.5) based on energy arguments. In Section 4.3 we introduce and compare a piecewise constant and a piecewise linear continuous-in-time adapted process, both of which coincide with the solution of the split-step scheme on the temporal grid. These auxiliary processes turn out to be useful tools and they already appear in the SDE setting, for example, in [17]. In Section 5 we analyse the error in the split-step scheme and in the backward Euler scheme and the conclusion of these results are summarised in Theorem 1.1 above.

2. PRELIMINARIES

Throughout the paper we will use various norms for linear operators on a Hilbert space. We denote by $\mathcal{L}(H)$, the space of bounded linear operators on H with the usual operator norm denoted by $\|\cdot\|$. If for a self-adjoint, positive semidefinite

operator $T: H \rightarrow H$, the sum

$$\mathrm{Tr} T := \sum_{k=1}^{\infty} \langle T e_k, e_k \rangle < \infty$$

for an orthonormal basis (ONB) $\{e_k\}_{k \in \mathbb{N}}$ of H , then we say that T is trace class. In this case $\mathrm{Tr} T$, the trace of T , is independent of the choice of the ONB. If for an operator $T: H \rightarrow H$, the sum

$$\|T\|_{\mathrm{HS}}^2 := \sum_{k=1}^{\infty} \|T e_k\|^2 < \infty$$

for an ONB $\{e_k\}_{k \in \mathbb{N}}$ of H , then we say that T is Hilbert–Schmidt and call $\|T\|_{\mathrm{HS}}$ the Hilbert–Schmidt norm of T . The Hilbert–Schmidt norm of T is independent of the choice of the ONB. We have the following well-known properties of the trace and Hilbert–Schmidt norms, see, for example, [6, Appendix C],

$$(2.1) \quad \|T\| \leq \|T\|_{\mathrm{HS}}, \quad \|TS\|_{\mathrm{HS}} \leq \|T\|_{\mathrm{HS}} \|S\|, \quad \|ST\|_{\mathrm{HS}} \leq \|S\| \|T\|_{\mathrm{HS}},$$

$$(2.2) \quad \mathrm{Tr} Q = \|Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 = \|T\|_{\mathrm{HS}}^2 = \|T^*\|_{\mathrm{HS}}^2, \quad \text{if } Q = TT^*.$$

Next, we introduce fractional order spaces and norms. It is well known that our assumptions on A and on the spatial domain \mathcal{D} imply the existence of a sequence of non-decreasing positive real numbers $\{\lambda_k\}_{k \geq 1}$ and an ONB $\{e_k\}_{k \geq 1}$ of H such that $A e_k = \lambda_k e_k$ and $\lim_{k \rightarrow \infty} \lambda_k = \infty$. We define the scalar product and norm of order $s \in \mathbf{R}$:

$$\langle v, w \rangle_s := \sum_{k=1}^{\infty} \lambda_k^s \langle v, e_k \rangle \langle w, e_k \rangle, \quad \|v\|_s^2 = \sum_{k=1}^{\infty} \lambda_k^s \langle v, e_k \rangle^2.$$

For $s \geq 0$ the fractional order space is defined by $\dot{H}^s = D(A^{s/2}) := \{v \in H : \|v\|_s < \infty\}$ and for $s < 0$ as the closure of H with respect to the $\|\cdot\|_s$ -norm. It is well-known that $\dot{H}^1 = H_0^1(\mathcal{D})$ and $\dot{H}^2 = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$.

We recall the fact that the semigroup $\{E(t)\}_{t \geq 0}$ generated by $-A$ is analytic because $-A$ is self-adjoint and negative definite, see, for example, [1, Example 3.7.5]. For such a semigroup it follows from the Spectral Theorem that for $\alpha \geq 0$,

$$(2.3) \quad \|A^\alpha E(t)v\| \leq C_\alpha t^{-\alpha} \|v\|, \quad C_\alpha = \sup\{\lambda^\alpha e^{-\lambda} : \lambda \geq 0\}. \quad t > 0.$$

We will also use the Burkholder–Davies–Gundy inequality for Itô-integrals of the form $\int_0^t \langle \eta(s), d\tilde{W}(s) \rangle$, where \tilde{W} is a \tilde{Q} -Wiener process. For this kind of integral, the Burkholder–Davies–Gundy inequality, [6, Lemma 7.2], takes the form

$$(2.4) \quad \mathbf{E} \sup_{t \in [0, T]} \left| \int_0^t \langle \eta(s), d\tilde{W}(s) \rangle \right|^p \leq C_p \mathbf{E} \left(\int_0^T \|\tilde{Q}^{\frac{1}{2}} \eta(s)\|^2 ds \right)^{\frac{p}{2}}, \quad p \geq 2.$$

Also, if Y is an H -valued Gaussian random variable with covariance operator \tilde{Q} , then, by [6, Corollary 2.17], we can bound its p -th moments via its covariance operator as

$$(2.5) \quad \mathbf{E} \|Y\|^{2p} \leq C_p (\mathbf{E} \|Y\|^2)^p = C_p (\mathrm{Tr} \tilde{Q})^p = \|\tilde{Q}^{\frac{1}{2}}\|_{\mathrm{HS}}^{2p}.$$

We will repeatedly apply this to the Itô integral $\int_s^t R dW(r)$, where R is a constant, possibly unbounded, operator on H and W is a Q -Wiener process. Then (2.5) reads

$$(2.6) \quad \mathbf{E} \left\| \int_s^t R dW(r) \right\|^{2p} \leq C_p (t-s)^p \|RQ^{1/2}\|_{\mathrm{HS}}^{2p}.$$

If $p = 1$, the inequality in (2.6) becomes an equality with $C = 1$. Moreover, the inequality

$$(2.7) \quad \left| \sum_{j=M}^K a_j \right|^p \leq |M - K + 1|^{p-1} \sum_{j=M}^K |a_j|^p$$

will be frequently utilised. This is a direct consequence of Hölder's inequality.

The existence of solutions to (1.1) and their regularity has been studied in, for example, [24] (variational solution). We summarise the results in the following theorem and we also refer to the discussion preceding and after [20, Proposition 3.1] for more details.

Theorem 2.1. *If $\|A^{\frac{1}{2}}Q^{\frac{1}{2}}\|_{\text{HS}} < \infty$ and $\mathbf{E}\|X_0\|_1^p < \infty$ for some $p \geq 2$, then there is a unique variational solution X of (1.1). Furthermore, there is $C_T > 0$ such that*

$$(2.8) \quad \mathbf{E} \sup_{t \in [0, T]} \|X(t)\|^p + \mathbf{E} \sup_{t \in [0, T]} \|X(t)\|_1^p \leq C_T.$$

In addition, X also solves the mild equation, that is, we have almost surely,

$$(2.9) \quad X(t) = E(t)X_0 - \int_0^t E(t-s)f(s) \, ds + \int_0^t E(t-s) \, dW(s), \quad t \in [0, T].$$

We shall also utilise the following versions of Gronwall's lemma in continuous and discrete time. For a proof of the former, see [7] and for the latter, see [9].

Lemma 2.2 (Generalised Gronwall lemma). *Let $u \in L_1([0, T], \mathbb{R})$ be non-negative. If $\alpha, C_1, C_2 > 0$ and*

$$u(t) \leq C_1 + C_2 \int_0^t (t-s)^{\alpha-1} u(s) \, ds, \quad 0 < t \leq T,$$

then there exists $C = C(T, C_2, \alpha) > 0$ such that

$$u(t) \leq C_1 C, \quad 0 \leq t \leq T.$$

Lemma 2.3 (Discrete Gronwall lemma). *Let $\{a_j\}_{j=0}^N$ be a real non-negative sequence. If, for some $C_1, C_2 > 0$,*

$$a_n \leq C_1 + C_2 \sum_{j=0}^{n-1} a_j, \quad 0 \leq n \leq N, \quad \text{then} \quad a_n \leq C_1 e^{C_2 n}, \quad 0 \leq n \leq N.$$

We will use the notation P_j to denote a positive polynomial of degree j , that is, $P_j(x) \geq 0$ whenever the argument $x \in \mathbf{R}^N$ has positive components.

3. REFORMULATION OF THE PROBLEM

For sufficiently small k_0 , to be made precise in Lemma 3.3, equations (1.3)–(1.5) are all uniquely solvable if $k < k_0$ and thus, for $j = 0, 1, \dots, N-1$,

$$(3.1) \quad \begin{aligned} X^0 &= X_0, \\ Y^j &= R_{\frac{k}{2}} X^j, \quad \text{where } R_{\frac{k}{2}} := (I + \frac{k}{2}A)^{-1}, \end{aligned}$$

$$(3.2) \quad Z^j = J_k Y^j, \quad \text{where } J_k(x) := (I + kf)^{-1}x,$$

$$(3.3) \quad X^{j+1} = R_{\frac{k}{2}} Z^j + R_{\frac{k}{2}} \Delta W^j.$$

We insert (3.1) into (3.2) to get $Z^j = J_k R_{\frac{k}{2}} X^j$. We then substitute this and (3.1) into (1.4), to get $Z^j = R_{\frac{k}{2}} X^j - k f(J_k R_{\frac{k}{2}} X^j)$. If this is substituted into (3.3), we arrive at

$$(3.4) \quad \begin{aligned} X^{j+1} &= R_{\frac{k}{2}}^2 X^j - k R_{\frac{k}{2}} f(J_k R_{\frac{k}{2}} X^j) + R_{\frac{k}{2}} \Delta W^j \\ &= X^j - k A(I + \frac{k}{4} A) R_{\frac{k}{2}}^2 X^j - k R_{\frac{k}{2}} f(J_k R_{\frac{k}{2}} X^j) + R_{\frac{k}{2}} \Delta W^j, \end{aligned}$$

where we used the identity $R_{\frac{k}{2}}^2 = I - k A(I + \frac{k}{4} A) R_{\frac{k}{2}}^2$. With the definitions

$$\begin{aligned} M_k &= (I + \frac{k}{4} A), & \mathcal{A}_k &= A M_k R_{\frac{k}{2}}^2, \\ f_k(\cdot) &= f(J_k \cdot), & F_k(\cdot) &= R_{\frac{k}{2}} f_k(R_{\frac{k}{2}} \cdot), \end{aligned}$$

we can write (3.4) in the form

$$X^{j+1} = X^j - k \mathcal{A}_k X^j - k F_k(X^j) + R_{\frac{k}{2}} \Delta W^j.$$

This may be viewed as the fully explicit Euler scheme, or the Euler–Maruyama scheme, for the equation

$$(3.5) \quad d\tilde{X}_k + \mathcal{A}_k \tilde{X}_k dt + F_k(\tilde{X}_k) dt = R_{\frac{k}{2}} dW.$$

Note that the operator $-\mathcal{A}_k$ is self-adjoint and negative definite and hence it is the generator of an analytic semigroup $E_k(t) = e^{-t\mathcal{A}_k}$, and (2.3) holds with A and E replaced by \mathcal{A}_k and E_k , respectively. As we shall see \mathcal{A}_k is also a bounded operator on H and F_k is Lipschitz continuous and thus (3.5) admits unique strong, weak, mild, and variational solutions, if also $\|R_{\frac{k}{2}} Q^{1/2}\|_{\text{HS}} < \infty$. Further, such solutions coincide, see [25, Appendix F]. In particular, (3.5) may be written in its mild form,

$$(3.6) \quad \tilde{X}_k(t) = E_k(t) X_0 - \int_0^t E_k(t-s) F_k(\tilde{X}_k(s)) ds + \int_0^t E_k(t-s) R_{\frac{k}{2}} dW(s)$$

or in its strong form

$$\tilde{X}_k(t) = - \int_0^t \mathcal{A}_k \tilde{X}_k(s) + F_k(\tilde{X}_k(s)) ds + \int_0^t R_{\frac{k}{2}} dW(s).$$

3.1. Properties of the new operators. The operators $R_{\frac{k}{2}}$, M_k , and \mathcal{A}_k introduced above are self-adjoint positive definite and they commute with A . In fact, M_k is coercive in the sense that

$$(3.7) \quad \|M_k x\| \geq C \|x\|, \quad x \in \dot{H}^2,$$

and the operators $R_{\frac{k}{2}}$, $M_k R_{\frac{k}{2}}$ satisfy

$$(3.8) \quad \|R_{\frac{k}{2}} x\| \leq \|x\|, \quad x \in H,$$

$$(3.9) \quad \frac{1}{2} \|x\| \leq \|M_k R_{\frac{k}{2}} x\| \leq \|x\|, \quad x \in H,$$

and hence, for \mathcal{A}_k ,

$$(3.10) \quad \frac{1}{2} \|A R_{\frac{k}{2}} x\| \leq \|\mathcal{A}_k x\| \leq \|A R_{\frac{k}{2}} x\|, \quad x \in H.$$

The operator $A R_{\frac{k}{2}}$ is (essentially) the Yosida approximation of A with the bounds

$$(3.11) \quad \|A R_{\frac{k}{2}} x\| \leq k^{-1} C \|x\|, \quad \|A R_{\frac{k}{2}} x\| \leq C \|x\|_2,$$

and the interpolated version

$$(3.12) \quad \|AR_{\frac{k}{2}}x\| \leq k^{-s/2}C\|x\|_{2-s}, \quad 0 \leq s \leq 2.$$

An immediate consequence is that

$$(3.13) \quad \|A^{\alpha/2}R_{\frac{k}{2}}x\| \leq k^{-s/2}C\|x\|_{\alpha-s}, \quad 0 \leq s \leq 2, \quad \alpha \in \mathbf{R}.$$

Note also that (3.10) and (3.11) show that, indeed, $\mathcal{A}_k \in \mathcal{B}(H)$. We will use (3.13), in particular, with $\alpha = s = 1$, that is,

$$(3.14) \quad \|A^{1/2}R_{\frac{k}{2}}x\| \leq k^{-1/2}C\|x\|.$$

If $k_1, k_2 \geq 0$ are two time-steps, then we have that

$$(3.15) \quad R_{\frac{k_1}{2}} - R_{\frac{k_2}{2}} = \frac{1}{2}(k_2 - k_1)AR_{\frac{k_1}{2}}R_{\frac{k_2}{2}}$$

and, assuming $k_1 \geq k_2$,

$$(3.16) \quad \|(R_{\frac{k_1}{2}} - R_{\frac{k_2}{2}})x\| \leq Ck_1^{s/2}\|R_{\frac{k_2}{2}}x\|_s \leq Ck_1^{s/2}\|x\|_s, \quad 0 \leq s \leq 2.$$

We note that if $A^{1/2}Q^{1/2}$ is Hilbert–Schmidt, then it is bounded according to (2.1) and also $A^{1/2}R_{\frac{k}{2}}Q^{1/2}$ is Hilbert–Schmidt, hence bounded. More precisely, the following result holds.

Lemma 3.1. *For all $k \geq 0$ we have that $\|Q^{1/2}A^{1/2}R_{\frac{k}{2}}\| \leq \|A^{1/2}Q^{1/2}\|_{\text{HS}}$.*

Proof. By (2.1) and (2.2) we have that

$$\begin{aligned} \|Q^{1/2}A^{1/2}R_{\frac{k}{2}}\| &\leq \|Q^{1/2}A^{1/2}R_{\frac{k}{2}}\|_{\text{HS}} = \|(Q^{1/2}A^{1/2}R_{\frac{k}{2}})^*\|_{\text{HS}} = \|R_{\frac{k}{2}}A^{1/2}Q^{1/2}\|_{\text{HS}} \\ &\leq \|R_{\frac{k}{2}}\| \|A^{1/2}Q^{1/2}\|_{\text{HS}} \leq \|A^{1/2}Q^{1/2}\|_{\text{HS}}. \end{aligned}$$

The second equality is motivated by the self-adjointness of all involved operators and the boundedness in H of $A^{1/2}Q^{1/2}$ and $R_{\frac{k}{2}}$. \square

We will make use of the following two deterministic error estimates.

Lemma 3.2. *Let $E(t)$ and $E_k(t)$ be the analytic semigroups generated by $-A$ and $-\mathcal{A}_k$, respectively. Then for $x \in \dot{H}^{s-r}$, with $0 \leq r \leq s \leq 2$, we have that*

$$(3.17) \quad \|(E(t) - E_k(t)R_{\frac{k}{2}})x\| \leq Ck^{\frac{s}{2}}t^{-\frac{r}{2}}\|x\|_{s-r}, \quad t > 0,$$

$$(3.18) \quad \|(E(t) - E_k(t))x\| \leq Ck^{\frac{s}{2}}\|x\|_s, \quad t > 0.$$

Proof. In view of the spectral calculus for A we consider the difference

$$\begin{aligned} F(t) &= e^{-\lambda t} - \frac{1}{1 + \frac{k\lambda}{2}} e^{-\lambda t \frac{1 + \frac{k\lambda}{2}}{(1 + \frac{k\lambda}{2})^2}} = e^{-\lambda t} - e^{-\ln(1 + \frac{k\lambda}{2}) - t\lambda(1 - k\lambda g(k\lambda))} \\ &= e^{-\lambda t} - e^{-\lambda t(1 - k\lambda g(k\lambda) + \frac{\ln(1 + \frac{k\lambda}{2})}{\lambda t})} \end{aligned}$$

with $g(x) = \frac{3+x}{4(1+\frac{x}{2})^2}$. We set $G(x) = xg(x)$ and $f(x) = G(x) - \frac{\ln(1+\frac{x}{2})}{\lambda t}$. An elementary calculation shows that $G'(x) > 0$ for $x \geq 0$, $f(0) = 0$, and $f'(x) \geq 0$ if $\lambda t \geq x \geq 2$.

Let first $\lambda t \geq \lambda k \geq 2$. Then

$$\begin{aligned} |F(t)| &= \left| \lambda t e^{-\lambda t} \int_{f(k\lambda)}^0 e^{\lambda t x} dx \right| \leq \lambda t e^{-\lambda t} f(k\lambda) e^{\lambda t f(k\lambda)} \\ &\leq \lambda t e^{-\lambda t} G(k\lambda) e^{\lambda t G(k\lambda)} \frac{1}{1 + \frac{k\lambda}{2}} \\ &= e^{-\lambda t(1 - k\lambda g(k\lambda))} (\lambda t)^{1 + \frac{r}{2}} \frac{k^{\frac{s}{2}}}{t^{\frac{r}{2}}} (k\lambda)^{1 - \frac{s}{2}} g(k\lambda) \frac{\lambda^{\frac{s-r}{2}}}{1 + \frac{k\lambda}{2}}. \end{aligned}$$

The function $h(x) = e^{-x\epsilon} x^{1+\alpha}$ is maximised at $x = \frac{1+\alpha}{\epsilon}$ and $h(\frac{1+\alpha}{\epsilon}) = C_\alpha \epsilon^{-(1+\alpha)}$ with C_α uniformly bounded, in α , on bounded subintervals of $[0, \infty)$. If we set $\epsilon = (1 - k\lambda g(k\lambda)) = \frac{1 + \frac{k\lambda}{2}}{(1 + \frac{k\lambda}{2})^2}$ and $\alpha = \frac{r}{2}$, then, for $k\lambda \geq 2$ and $0 \leq r \leq s \leq 2$,

$$\begin{aligned} |F(t)| &\leq C \frac{(1 + \frac{k\lambda}{2})^{2+r}}{(1 + \frac{k\lambda}{2})^{1+\frac{r}{2}}} \frac{1}{1 + \frac{k\lambda}{2}} \frac{k^{\frac{s}{2}}}{t^{\frac{r}{2}}} (k\lambda)^{1 - \frac{s}{2}} g(k\lambda) \lambda^{\frac{s-r}{2}} \\ &\leq C (k\lambda)^{\frac{r}{2}} \frac{k^{\frac{s}{2}}}{t^{\frac{r}{2}}} (k\lambda)^{1 - \frac{s}{2}} g(k\lambda) \lambda^{\frac{s-r}{2}} \leq C \frac{k^{\frac{s}{2}}}{t^{\frac{r}{2}}} \lambda^{\frac{s-r}{2}}. \end{aligned}$$

In the last inequality we used the fact that the function $H(x) = x^\beta g(x)$ is bounded if $0 \leq \beta \leq 1$.

Let now $\lambda k \geq 2$ and $\lambda t < \lambda k$; that is, $1 < \frac{k}{t}$. Then, for $0 \leq r \leq s \leq 2$,

$$|F(t)| \leq 2 \leq 2(\lambda k)^{\frac{s-r}{2}} \left(\frac{k}{t} \right)^{\frac{r}{2}} = 2 \frac{k^{\frac{s}{2}}}{t^{\frac{r}{2}}} \lambda^{\frac{s-r}{2}}.$$

Finally, let $\lambda k < 2$. Then, as G is monotone, we have $G(x) = xg(x) \leq G(2) = \frac{5}{8} < 1$ for $x \in [0, 2]$. We compute

$$\begin{aligned} F(t) &= e^{-\lambda t} - \frac{1}{1 + \frac{k\lambda}{2}} e^{-\lambda t} + \frac{1}{1 + \frac{k\lambda}{2}} (e^{-\lambda t} (1 - e^{\lambda t k \lambda g(k\lambda)})) \\ &= \frac{\lambda k}{2} \frac{1}{1 + \frac{k\lambda}{2}} e^{-\lambda t} + \frac{1}{1 + \frac{k\lambda}{2}} (e^{-\lambda t} (1 - e^{\lambda t k \lambda g(k\lambda)})) =: F_1(t) + F_2(t). \end{aligned}$$

For $0 \leq r \leq s \leq 2$, it follows that

$$F_1(t) = \frac{\frac{1}{2} k^{\frac{s}{2}} (\lambda t)^{\frac{r}{2}} e^{-\lambda t} (\lambda k)^{1 - \frac{s}{2}} \lambda^{\frac{s-r}{2}}}{t^{\frac{r}{2}} (1 + \frac{k\lambda}{2})} \leq C \frac{k^{\frac{s}{2}}}{t^{\frac{r}{2}}} \lambda^{\frac{s-r}{2}}.$$

For F_2 we write

$$F_2(t) = \frac{1}{1 + \frac{k\lambda}{2}} e^{-t\lambda} t \lambda \int_{k\lambda g(k\lambda)}^0 e^{t\lambda x} dx,$$

and hence, for $\lambda k < 2$ and $0 \leq r \leq s \leq 2$,

$$\begin{aligned} |F_2(t)| &\leq (e^{-t\lambda} t \lambda) (k\lambda g(k\lambda)) (e^{\frac{5}{8} t \lambda}) \\ &= (t\lambda)^{1 + \frac{r}{2}} e^{-\frac{3}{8} t \lambda} \frac{k^{\frac{s}{2}}}{t^{\frac{r}{2}}} (k\lambda)^{1 - \frac{s}{2}} g(k\lambda) \lambda^{\frac{s-r}{2}} \leq C \frac{k^{\frac{s}{2}}}{t^{\frac{r}{2}}} \lambda^{\frac{s-r}{2}}. \end{aligned}$$

This finishes the proof of (3.17).

For (3.18) we have, by (3.17) with $r = 0$ and (3.16) with $k_2 = 0$, that

$$\|(E(t) - E_k(t))x\| \leq \|(E(t) - E_k(t)R_{\frac{k}{2}})x\| + \|E_k(t)\|(R_{\frac{k}{2}}x - x)\| \leq C k^{\frac{s}{2}} \|x\|_s.$$

□

We now turn to the non-linear operator $f: D_f \rightarrow H$ and its approximations. Since f is a third degree polynomial and the spatial dimension $d \leq 3$, we have, by Sobolev's inequality, that $\dot{H}^1 \subset L_6(\mathcal{D}) \subset D_f$. We shall see that if $2k\beta^2 < 1$, then the equation

$$(3.19) \quad z + kf(z) = x, \quad x \in H,$$

has a unique solution and we may define an inverse operator

$$(3.20) \quad z = J_k(x) := (I + kf)^{-1}(x).$$

This is the resolvent operator of f . The range of J_k is contained in D_f and thus the Yosida approximation of f , that is,

$$(3.21) \quad f_k(x) := f(J_k(x)), \quad x \in H,$$

is also well defined. Furthermore, the resolvent operator $R_{\frac{k}{2}}$ of A is bounded from $\dot{H}^s \rightarrow \dot{H}^{s+2}$ for any s and thus

$$(3.22) \quad F_k(x) := R_{\frac{k}{2}} f_k(R_{\frac{k}{2}} x)$$

is well defined, too.

We shall now prove the non-trivial of these claims and further properties of the non-linear operators that we need in the sequel.

Lemma 3.3. *Let f be as described above and let J_k , f_k , and F_k be as in (3.20), (3.21), and (3.22), respectively. Also, assume that $k \leq k_0$ with $2k_0\beta^2 < 1$.*

- (1) *The equation (3.19) has a unique solution $z \in L_6(\mathcal{D})$ for every $x \in H$; that is, $J_k: H \rightarrow H$ is well defined. Furthermore, $f(z) \in H$ so that $f_k: H \rightarrow H$ is also well defined. If, in addition, $x \in \dot{H}^1$, then also $z, f(z) \in \dot{H}^1$.*
- (2) *The operator $J_k: H \rightarrow H$ is Lipschitz continuous and obeys a linear growth condition on \dot{H}^1 with constants that are bounded as $k \rightarrow 0$; more precisely,*

$$(3.23) \quad \|J_k(x) - J_k(y)\| \leq C(k)\|x - y\|,$$

$$(3.24) \quad \|J_k(x)\|_1 \leq C(k)\|x\|_1$$

with $C(k) = \frac{1}{\sqrt{1-2k\beta^2}}$.

- (3) *The operators $f_k, F_k: H \rightarrow H$ are Lipschitz continuous with Lipschitz constant $1/k$. Furthermore, they have the same one-sided Lipschitz and dissipativity properties as f ; that is, (1.6), (1.9), and (1.10) hold true with f replaced by f_k or F_k and β^2 replaced by $\beta^2/(1-2k\beta^2)$.*
- (4) *It holds that*

$$(3.25) \quad x - J_k(x) = kf_k(x), \quad x \in H.$$

- (5) *It also holds that*

$$(3.26) \quad \begin{aligned} \langle F_\alpha(x) - F_\beta(y), x - y \rangle &= \langle f_\alpha(R_{\frac{\alpha}{2}}x) - f_\beta(R_{\frac{\beta}{2}}y), R_{\frac{\alpha}{2}}x - R_{\frac{\beta}{2}}y \rangle \\ &+ \langle f_\alpha(R_{\frac{\alpha}{2}}x), (R_{\frac{\alpha}{2}} - R_{\frac{\beta}{2}})y \rangle \\ &+ \langle f_\beta(R_{\frac{\beta}{2}}y), (R_{\frac{\alpha}{2}} - R_{\frac{\beta}{2}})x \rangle, \quad x, y \in H. \end{aligned}$$

Proof. To prove Item (1) we consider the function $g(t) = t + kf(t) - s$ for arbitrary but fixed $s \in \mathbb{R}$. We have that $g'(t) = 1 + k3t^2 - k\beta^2$, so if $k\beta^2 < 1$, then g is monotone and since it is a cubic polynomial the equation $g(t) = 0$ has exactly one

solution $t = t(s)$. It follows that if x is a continuous function on \mathcal{D} , equation (3.19) has a unique solution z that is also continuous since f is continuous function on \mathbb{R} .

If $x \in H = L_2(\mathcal{D})$ is arbitrary, then we may approximate it by continuous functions. Let $\{x_n\}_{n=1}^\infty$ be a sequence of continuous functions such that $\lim_{n \rightarrow \infty} x_n = x$ in H and let $\{z_n\}_{n=1}^\infty$ be the corresponding solutions to (3.19) with x replaced by x_n . Using the particular form $f(x) = x^3 - \beta^2 x$, we shall show that $z_n \rightarrow z$ in $L_6(\mathcal{D})$ and that this implies $z_n^3 \rightarrow z^3$ in H . This means that $f(z_n) \rightarrow f(z)$ in H , which in turn implies the solvability of (3.19) as well as the fact that f_k is a well defined operator on H .

To do this we note that $\langle f(x), x \rangle = \|x\|_{L_4(\mathcal{D})}^4 - \beta^2 \|x\|^2$ and $\|f(x)\|^2 = \|x\|_{L_6(\mathcal{D})}^6 - 2\beta^2 \|x\|_{L_4(\mathcal{D})}^4 + \beta^4 \|x\|^2$. Taking the squared norm of both sides of (3.19) we therefore see that, if $2k\beta^2 < 1$, then

$$\begin{aligned}
 \|x_n\|^2 &= \|z_n\|^2 + 2k\langle f(z_n), z_n \rangle + k^2 \|f(z_n)\|^2 \\
 &= \|z_n\|^2 + 2k(\|z_n\|_{L_4(\mathcal{D})}^4 - \beta^2 \|z_n\|^2) \\
 &\quad + k^2(\|z_n\|_{L_6(\mathcal{D})}^6 - 2\beta^2 \|z_n\|_{L_4(\mathcal{D})}^4 + \beta^4 \|z_n\|^2) \\
 (3.27) \quad &\geq (1 - 2k\beta^2)(\|z_n\|^2 + k^2 \|z_n\|_{L_6(\mathcal{D})}^6) \\
 &\geq (1 - 2k_0\beta^2)(\|z_n\|^2 + k^2 \|z_n\|_{L_6(\mathcal{D})}^6).
 \end{aligned}$$

Since the sequence $\{x_n\}_{n=1}^\infty$ is uniformly bounded in H it follows that $\{z_n\}_{n=1}^\infty$ is uniformly bounded in $L_6(\mathcal{D})$.

To proceed we claim that $\{z_n^3\}_{n=1}^\infty$ is a Cauchy sequence in H . To see this, take the squared norm of both sides of the identity

$$(3.28) \quad z_i - z_j + k(f(z_i) - f(z_j)) = x_i - x_j$$

and use that

$$\|f(z_i) - f(z_j)\|^2 = \|z_i^3 - z_j^3\|^2 - 2\beta^2 \langle f(z_i) - f(z_j), z_i - z_j \rangle + \beta^4 \|z_i - z_j\|^2$$

to compute

$$\begin{aligned}
 \|x_i - x_j\|^2 &= \|z_i - z_j\|^2 + 2k\langle f(z_i) - f(z_j), z_i - z_j \rangle + k^2 \|f(z_i) - f(z_j)\|^2 \\
 (3.29) \quad &= \|z_i - z_j\|^2 + 2k(1 - k\beta^2) \langle f(z_i) - f(z_j), z_i - z_j \rangle \\
 &\quad + k^2 \|z_i^3 - z_j^3\|^2 + k^2 \beta^4 \|z_i - z_j\|^2.
 \end{aligned}$$

Thus, if $2k\beta^2 < 1$, then (1.6) implies that

$$\begin{aligned}
 \|x_i - x_j\|^2 &\geq \|z_i - z_j\|^2 - 2k(1 - k\beta^2)\beta^2 \|z_i - z_j\|^2 + k^2 \|z_i^3 - z_j^3\|^2 \\
 (3.30) \quad &\geq \|z_i - z_j\|^2 - 2k\beta^2 \|z_i - z_j\|^2 + k^2 \|z_i^3 - z_j^3\|^2 \\
 &\geq (1 - 2k\beta^2) \|z_i - z_j\|^2 + k^2 \|z_i^3 - z_j^3\|^2.
 \end{aligned}$$

Thus, $\{z_n^3\}_{n=1}^\infty$ is Cauchy in H .

It is not hard to see that if $x, y \in L_6(\mathcal{D})$, then

$$(3.31) \quad \|x - y\|_{L_6(\mathcal{D})}^6 \leq C \|x^3 - y^3\|^2$$

for a sufficiently large positive number C . Indeed, note that (3.31) is equivalent to

$$0 \leq \int_{\mathcal{D}} C(x^3 - y^3)^2 - (x - y)^6 \, d\xi = \int_{\mathcal{D}} (x - y)^2 (C(x^2 + xy + y^2)^2 - (x - y)^4) \, d\xi.$$

Thus, we need to find a C such that $P_C(t, s) := C(t^2 + ts + t^2)^2 - (t - s)^4 \geq 0$ for all real t and s . We have that

$$P_C(t, s) = (C - 1)(t^4 + s^4) + 2(C + 2)(ts)(t^2 + s^2) + 3(C - 2)(t^2 s^2).$$

If $ts \geq 0$, then $C > 2$ suffices. Assume $C > 2$ and $ts < 0$. Then

$$P_C(t, s) \geq (C - 1)(t^4 + s^4) - 2(C + 2)(\epsilon t^2 s^2 + \frac{1}{2\epsilon}(t^4 + s^4)) + 3(C - 2)t^2 s^2.$$

Take $\epsilon = \frac{3(C-2)}{2(C+2)}$, so that $\frac{1}{2\epsilon} = \frac{C+2}{3(C-2)}$. Then

$$P_C(t, s) \geq \left(C - 1 - \frac{2}{3} \frac{(C+2)^2}{(C-2)}\right)(t^4 + s^4).$$

We must find $C > 2$ such that

$$C - 1 - \frac{2}{3} \frac{(C+2)^2}{C-2} \geq 0.$$

Under the current restriction on C this holds if $C^2 - 17C - 2 \geq 0$. This is true for large C and (3.31) is proved.

By combining (3.30) and (3.31), we see that $\{z_n^3\}_{n=1}^\infty$ is Cauchy in $L_6(\mathcal{D})$. Thus, $z_n \rightarrow z$ for some z in $L_6(\mathcal{D})$. It remains to show that $z_n^3 \rightarrow z^3$ in H . But

$$\begin{aligned} \|z_n^3 - z^3\|^2 &= \|(z_n^2 + z_n z + z^2)(z_n - z)\|^2 \leq \|z_n^2 + z_n z + z^2\|_{L_3(\mathcal{D})}^2 \|z_n - z\|_{L_6(\mathcal{D})}^2 \\ &\leq C(\|z_n\|_{L_6(\mathcal{D})}^4 + \|z\|_{L_6(\mathcal{D})}^4) \|z_n - z\|_{L_6(\mathcal{D})}^2. \end{aligned}$$

Since $z \in L_6(\mathcal{D})$ and, by (3.27), the sequence $\{z_n\}_{n=1}^\infty$ is bounded in $L_6(\mathcal{D})$ and $z_n \rightarrow z$ in $L_6(\mathcal{D})$ the assertion is true. We have thus shown that there is a solution $z \in L_6(\mathcal{D})$ to (3.19) for every $x \in H$. Uniqueness follows from (3.30).

If $x \in \dot{H}^1$, then take the squared \dot{H}^1 -norm of (3.19) and use (1.10) to get

$$\begin{aligned} \|x\|_1^2 &= \|z\|_1^2 + 2k\langle f(z), z \rangle_1 + k^2\|f(z)\|_1^2 \\ &\geq \|z\|_1^2 - 2k\beta^2\|z\|_1^2 + k^2\|f(z)\|_1^2. \end{aligned} \tag{3.32}$$

Since $2k\beta^2 < 1$, we have that $z, f(z) \in \dot{H}^1$ if $x \in \dot{H}^1$. We have proved Item (1).

We turn to Item (2). Lipschitz continuity of J_k in H follows from (3.30), since

$$\|J_k(x_i) - J_k(x_j)\| = \|z_i - z_j\| \leq \frac{1}{\sqrt{1 - 2k\beta^2}} \|x_i - x_j\|.$$

The linear growth bound (3.24) in \dot{H}^1 follows from (3.32).

To prove Item (3), we use the first equality in (3.29) and (1.6) to conclude that if $2k\beta^2 < 1$, then

$$\|x_i - x_j\| \geq k\|f(z_i) - f(z_j)\|.$$

That is the claimed Lipschitz continuity of f_k in H . The claim about F_k follows immediately from this and the contraction property (3.8) of $R_{\frac{k}{2}}$.

For the one-sided Lipschitz conditions, we use (3.28), (1.6), and the result in Item (2). Indeed, for f_k ,

$$\begin{aligned} \langle f(z_1) - f(z_2), x_1 - x_2 \rangle &= \langle f(z_1) - f(z_2), z_1 - z_2 + k(f(z_1) - f(z_2)) \rangle \\ &\geq -\beta^2\|z_1 - z_2\|^2 + k\|f(z_1) - f(z_2)\|^2 \geq -\frac{\beta^2}{1 - 2k\beta^2}\|x_1 - x_2\|^2. \end{aligned} \tag{3.33}$$

For F_k , from (3.33) and the contraction property (3.8) of $R_{\frac{k}{2}}$, we have that

$$\begin{aligned} \langle F_k(x_1) - F_k(x_2), x_1 - x_2 \rangle &= \langle R_{\frac{k}{2}}(f_k(R_{\frac{k}{2}}x_1) - f_k(R_{\frac{k}{2}}x_2)), x_1 - x_2 \rangle \\ &= \langle f_k(R_{\frac{k}{2}}x_1) - f_k(R_{\frac{k}{2}}x_2), R_{\frac{k}{2}}x_1 - R_{\frac{k}{2}}x_2 \rangle \\ &\geq -\frac{\beta^2}{1-2k\beta^2} \|R_{\frac{k}{2}}x_1 - R_{\frac{k}{2}}x_2\| \geq -\frac{\beta^2}{1-2k\beta^2} \|x_1 - x_2\|. \end{aligned}$$

The fact that (1.9) holds for f_k and F_k follows from the one-sided Lipschitz conditions and the fact that $f_k(0) = F_k(0) = 0$.

To see that (1.10) holds for f_k (with a different constant) we use (1.10) for f , the definitions of z, f_k , and J_k as well as the linear growth (3.24) of J_k on \dot{H}^1 to conclude that

$$\begin{aligned} \langle f_k(x), x \rangle_1 &= \langle f(J_k(x)), z + kf(z) \rangle_1 = \langle f(z), z + kf(z) \rangle_1 = \langle f(z), z \rangle_1 + k\|f(z)\|_1^2 \\ &\geq -\beta^2 \|z\|_1^2 = -\beta^2 \|J_k(x)\|_1^2 \geq -\frac{\beta^2}{1-2k\beta^2} \|x\|_1^2. \end{aligned}$$

Therefore, as $R_{\frac{k}{2}}$ is also a contraction on \dot{H}^1 , it follows that

$$\begin{aligned} \langle F_k(x), x \rangle_1 &= \langle R_{\frac{k}{2}}f_k(R_{\frac{k}{2}}x), x \rangle_1 = \langle f_k(R_{\frac{k}{2}}x), R_{\frac{k}{2}}x \rangle_1 \\ &\geq -\frac{\beta^2}{1-2k\beta^2} \|R_{\frac{k}{2}}x\|_1^2 \geq -\frac{\beta^2}{1-2k\beta^2} \|x\|_1^2. \end{aligned}$$

The statement in Item (4) is only a rearrangement of (3.19) using $z = J_k(x)$.

To prove Item (5) we first note that

$$\begin{aligned} \langle F_\alpha(x) - F_\beta(y), x - y \rangle &= \langle F_\alpha(x), x - y \rangle - \langle F_\beta(y), x - y \rangle \\ &= \langle f_\alpha(R_\alpha x), R_\alpha(x - y) \rangle - \langle f_\beta(R_\beta y), R_\beta(x - y) \rangle. \end{aligned}$$

The statement follows by adding and subtracting the quantities $\langle f_\alpha(R_\alpha x), R_\beta y \rangle$ and $\langle f_\beta(R_\beta y), R_\alpha x \rangle$ and rearranging the terms. \square

We end this section by making the useful observation that, in view of (3.25) and the fact that $R_{\frac{k}{2}}x$ solves $(I + \frac{k}{2}A)y = x$, we have

$$(3.34) \quad x - J_k(R_{\frac{k}{2}}x) = x - R_{\frac{k}{2}}x + R_{\frac{k}{2}}x - J_k(R_{\frac{k}{2}}x) = \frac{k}{2}AR_{\frac{k}{2}}x + kf_k(R_{\frac{k}{2}}x).$$

Throughout the rest of the paper we will often write J_kx for $J_k(x)$ in order to increase readability of complicated formulae even though J_k is non-linear.

4. MOMENT BOUNDS ON SOLUTIONS

Various moment bounds on \tilde{X}_k and X^j are crucial for our convergence analysis. In order to prove these we assume moment bounds on the \dot{H}^1 -norm of the initial value and some spatial smoothness of the noise.

Assumption 4.1.

- (1) It holds that $k \leq k_0$ with $2k_0\beta^2 < 1$.
- (2) For the covariance operator Q of the Wiener process W , the inequality $\|A^{1/2}Q^{1/2}\|_{\text{HS}} < \infty$ holds.
- (3) For some $q \geq 1$, the q^{th} moment of the \dot{H}^1 -norm of the initial value X_0 is finite; that is $\mathbf{E}\|X_0\|_1^q < \infty$.

4.1. Bounds on \tilde{X}_k .

Lemma 4.2. *Let $\tilde{X}_k(t)$ be the solution of (3.5) and, for some $p \geq 1$, let Assumption 4.1 hold with $q = 2p$. Then there is $C_1 = C_1(k_0, T, p, \|A^{1/2}Q^{1/2}\|_{\text{HS}}, \mathbf{E}\|X_0\|_1^{2p})$ such that*

$$\mathbf{E}\left(\sup_{0 \leq t \leq T} \|\tilde{X}_k(t)\|_1^{2p}\right) \leq C_1$$

and $C_2 = C_2(k_0, T, \|A^{1/2}Q^{1/2}\|_{\text{HS}}, \mathbf{E}\|X_0\|_1^2)$ such that

$$\mathbf{E} \int_0^T \|\mathcal{A}_k^{1/2} \tilde{X}_k(s)\|_1^2 ds \leq C_2.$$

Proof. Since the functional $\|\cdot\|_1^2$ is not continuous on H , we can not apply Itô's formula directly. But note, first formally, that

$$\begin{aligned} \tilde{Y}(t) &:= A^{1/2} \tilde{X}_k(t) = E_k(t) A^{1/2} X_0 - \int_0^t E_k(t-s) A^{1/2} F_k(\tilde{X}_k(s)) ds \\ &\quad + \int_0^t E_k(t-s) A^{1/2} R_{\frac{k}{2}} dW(s) \\ (4.1) \quad &= E_k(t) A^{1/2} X_0 - \int_0^t E_k(t-s) A^{1/2} F_k(A^{-1/2} \tilde{Y}(s)) ds \\ &\quad + \int_0^t E_k(t-s) A^{1/2} R_{\frac{k}{2}} dW(s). \end{aligned}$$

Thus $\tilde{Y}(t)$ is the mild solution of

$$d\tilde{Y} + (\mathcal{A}_k \tilde{Y} + A^{1/2} F_k(A^{-1/2} \tilde{Y})) dt = A^{1/2} R_{\frac{k}{2}} dW, \quad t \in (0, T); \quad \tilde{Y}(0) = A^{1/2} X_0.$$

Since \mathcal{A}_k is bounded, $A^{1/2} F_k(A^{-1/2} \cdot)$ is globally Lipschitz, and $A^{1/2} R_{\frac{k}{2}} Q^{1/2}$ is a Hilbert–Schmidt operator, the mild solution exists and it equals the strong solution. Reverting the computation in (4.1) one finds that that $A^{-1/2} \tilde{Y}(t)$ solves (3.6). Hence, it does indeed hold that $\tilde{Y}(t) = A^{1/2} \tilde{X}_k(t)$.

Since $\frac{1}{2} \|\cdot\|^2$ is continuous on H , we have, almost surely ([2, Theorem 2.1]),

$$\begin{aligned} \frac{1}{2} \|\tilde{Y}(t)\|^2 &= \frac{1}{2} \|A^{1/2} X_0\|^2 + \int_0^t \langle \tilde{Y}(s), d\tilde{Y}(s) \rangle \\ &\quad + \frac{1}{2} \int_0^t \text{Tr} (A^{1/2} R_{\frac{k}{2}} Q^{1/2} (A^{1/2} R_{\frac{k}{2}} Q^{1/2})^*) ds \\ &= \frac{1}{2} \|A^{1/2} X_0\|^2 - \int_0^t \langle \tilde{Y}(s), \mathcal{A}_k \tilde{Y}(s) + A^{1/2} F_k(A^{-1/2} \tilde{Y}(s)) \rangle ds \\ &\quad + \int_0^t \langle \tilde{Y}(s), A^{1/2} R_{\frac{k}{2}} dW(s) \rangle \\ &\quad + \frac{1}{2} \int_0^t \text{Tr} (A^{1/2} R_{\frac{k}{2}} Q^{1/2} (A^{1/2} R_{\frac{k}{2}} Q^{1/2})^*) ds. \end{aligned}$$

Now, $\langle \tilde{Y}(s), \mathcal{A}_k \tilde{Y}(s) \rangle = \|\mathcal{A}_k^{1/2} \tilde{Y}(s)\|^2$ and

$$\begin{aligned} \langle \tilde{Y}(s), A^{1/2} F_k(A^{-1/2} \tilde{Y}(s)) \rangle &= \langle A^{1/2} A^{-1/2} \tilde{Y}(s), A^{1/2} F_k(A^{-1/2} \tilde{Y}(s)) \rangle \\ &= \langle A^{-1/2} \tilde{Y}(s), F_k(A^{-1/2} \tilde{Y}(s)) \rangle_1 \geq -C(k) \|A^{-1/2} \tilde{Y}(s)\|_1^2 = -C(k) \|\tilde{Y}(s)\|^2, \end{aligned}$$

so that

$$(4.2) \quad \begin{aligned} \frac{1}{2} \|\tilde{Y}(t)\|^2 + \int_0^t \|\mathcal{A}_k^{1/2} \tilde{Y}(s)\|^2 ds &\leq \frac{1}{2} \|A^{1/2} X_0\|^2 + C(k) \int_0^t \|\tilde{Y}(s)\|^2 ds \\ &+ \int_0^t \langle \tilde{Y}(s), A^{1/2} R_{\frac{k}{2}} dW(s) \rangle + t \|A^{1/2} R_{\frac{k}{2}} Q^{1/2}\|_{\text{HS}}^2. \end{aligned}$$

If we drop the integral on the left side and raise the remaining parts to the power $p \geq 1$, take the supremum in time and then the expectation, we find, with the aid of (2.7) and Hölder's inequality, that

$$(4.3) \quad \begin{aligned} \mathbf{E} \sup_{0 \leq t \leq T} \|\tilde{Y}(t)\|^{2p} &\leq C(k, p, T) \left(\mathbf{E} \|A^{1/2} X_0\|^{2p} + \mathbf{E} \int_0^T \|\tilde{Y}(s)\|^{2p} ds \right. \\ &\left. + \mathbf{E} \sup_{0 \leq t \leq T} \left| \int_0^t \langle \tilde{Y}(s), A^{1/2} R_{\frac{k}{2}} dW(s) \rangle \right|^p + \|A^{1/2} R_{\frac{k}{2}} Q^{1/2}\|_{\text{HS}}^{2p} \right). \end{aligned}$$

Using first that $|ab| \leq \frac{1}{2}(a^2 + b^2)$ and then the Burkholder–Davies–Gundy inequality (2.4), we obtain

$$(4.4) \quad \begin{aligned} \mathbf{E} \sup_{0 \leq t \leq T} \left| \int_0^t \langle \tilde{Y}(s), A^{1/2} R_{\frac{k}{2}} dW(s) \rangle \right|^p \\ \leq C(1 + \mathbf{E} \sup_{0 \leq t \leq T} \left| \int_0^t \langle \tilde{Y}(s), A^{1/2} R_{\frac{k}{2}} dW(s) \rangle \right|^{2p}) \\ \leq C(p)(1 + \mathbf{E} \left(\int_0^T \|Q^{1/2} A^{1/2} R_{\frac{k}{2}} \tilde{Y}(s)\|^2 ds \right)^p). \end{aligned}$$

As, by Lemma 3.1, we have that

$$\|Q^{1/2} A^{1/2} R_{\frac{k}{2}} \tilde{Y}(s)\| \leq \|Q^{1/2} A^{1/2} R_{\frac{k}{2}}\| \|\tilde{Y}(s)\| \leq \|A^{1/2} Q^{1/2}\|_{\text{HS}} \|\tilde{Y}(s)\|,$$

we can use Hölder's inequality to obtain

$$(4.5) \quad \begin{aligned} \mathbf{E} \sup_{0 \leq t \leq T} \left| \int_0^t \langle \tilde{Y}(s), A^{1/2} R_{\frac{k}{2}} dW(s) \rangle \right|^p \\ \leq C(p, T, \|A^{1/2} Q^{1/2}\|_{\text{HS}}) \left(1 + \mathbf{E} \int_0^T \|\tilde{Y}(s)\|^{2p} ds \right). \end{aligned}$$

Since $\mathbf{E} \int_0^T \|\tilde{Y}(s)\|^{2p} ds = \int_0^T \mathbf{E} \|\tilde{Y}(s)\|^{2p} ds \leq \int_0^T \mathbf{E} \sup_{0 \leq r \leq s} \|\tilde{Y}(r)\|^{2p} ds$, we deduce from (4.5) and (4.3) that

$$\begin{aligned} \mathbf{E} \sup_{0 \leq t \leq T} \|\tilde{Y}(t)\|^{2p} &\leq C(k_0, p, T, \|A^{1/2} Q^{1/2}\|_{\text{HS}}, \mathbf{E} \|X_0\|_1^{2p}) \\ &\times \left(1 + \int_0^T \mathbf{E} \sup_{0 \leq r \leq s} \|\tilde{Y}(r)\|^{2p} ds \right). \end{aligned}$$

By Gronwall's lemma, there is $C_1 = C_1(k_0, p, T, \|A^{1/2} Q^{1/2}\|_{\text{HS}}, \mathbf{E} \|X_0\|_1^{2p})$ such that

$$(4.6) \quad \mathbf{E} \sup_{0 \leq t \leq T} \|\tilde{Y}(t)\|^{2p} \leq C_1.$$

Returning to (4.2), we can find $C_2 = C_2(k_0, T, \|A^{1/2} Q^{1/2}\|_{\text{HS}}, \mathbf{E} \|X_0\|_1^2)$ such that

$$\mathbf{E} \int_0^t \|\mathcal{A}_k^{1/2} \tilde{Y}(s)\|^2 ds \leq C_2,$$

by using that the expectation of the Itô integral vanishes and (4.6) with $p = 1$. \square

Corollary 4.3. *Let $\tilde{X}_k(t)$ be the solution of (3.5) and, for some $p \geq 1$, let Assumption 4.1 hold with $q = 6p$. Then there is $C = C(k_0, p, T, \|A^{1/2}Q^{1/2}\|_{\text{HS}}, \mathbf{E}\|X_0\|_1^{6p})$ such that*

$$\mathbf{E} \int_0^T \|f_k(R_{\frac{k}{2}}\tilde{X}_k(s))\|^{2p} ds \leq C.$$

Proof. Since, by (1.8) and (3.24),

$$\|f_k(R_{\frac{k}{2}}\tilde{X}_k(s))\| \leq C(\|J_k(R_{\frac{k}{2}}\tilde{X}_k(s))\|_1^3 + 1) \leq C(k)(\|\tilde{X}_k(s)\|_1^3 + 1),$$

it holds that

$$\begin{aligned} \mathbf{E} \int_0^T \|f_k(R_{\frac{k}{2}}\tilde{X}_k(s))\|^{2p} ds &\leq C(k, p) \mathbf{E} \int_0^T (\|\tilde{X}_k(s)\|_1^{6p} + 1) ds \\ &\leq C(k, p, T) \mathbf{E} \sup_{0 \leq s \leq T} (\|\tilde{X}_k(s)\|_1^{6p} + 1) \end{aligned}$$

and the last expression is bounded by Lemma 4.2. \square

4.2. Moment bounds for the solution of the split-step scheme. We shall now prove similar results for the solution of (1.2)–(1.5). We note that under Assumption 4.1 with $q = 2$, X_0 and ΔW^j belong to \dot{H}^1 almost surely. Since $R_{\frac{k}{2}}$ and F_k map \dot{H}^1 into itself the upcoming calculations are motivated almost surely.

To begin, take \dot{H}^1 -inner products with Y^j in (1.3) and Z^j in (1.4) to get

$$\begin{aligned} \|Y^j\|_1^2 + \frac{k}{2}\|Y^j\|_2^2 &= \langle X^j, Y^j \rangle_1 \leq \frac{1}{2}\|X^j\|_1^2 + \frac{1}{2}\|Y^j\|_1^2, \\ \|Z^j\|_1^2 + k\langle f(Z^j), Z^j \rangle_1 &= \langle Y^j, Z^j \rangle_1 \leq \frac{1}{2}\|Y^j\|_1^2 + \frac{1}{2}\|Z^j\|_1^2. \end{aligned}$$

We invoke (1.10), assume that $k \leq k_0$ with $2k_0\beta^2 < 1$, and rearrange to get

$$\|Y^j\|_1^2 + k\|Y^j\|_2^2 \leq \|X^j\|_1^2, \quad \|Z^j\|_1^2 \leq \left(1 + \frac{2k\beta^2}{1 - 2k\beta^2}\right)\|Y^j\|_1^2.$$

Thus,

$$(4.7) \quad \|Z^j\|_1^2 \leq (1 + kC(k_0))\|X^j\|_1^2.$$

Further, by taking the squared \dot{H}^1 -norm of (1.5), we get

$$(4.8) \quad \|X^{j+1}\|_1^2 + k\|X^{j+1}\|_2^2 + \frac{k^2}{4}\|X^{j+1}\|_3^2 = \|Z^j\|_1^2 + 2\langle Z^j, \Delta W^j \rangle_1 + \|\Delta W^j\|_1^2.$$

By inserting the bound on $\|Z^j\|_1^2$ from (4.7) into (4.8), we get

$$\|X^{j+1}\|_1^2 + k\|X^{j+1}\|_2^2 \leq \|X^j\|_1^2 + kC\|X^j\|_1^2 + 2\langle Z^j, \Delta W^j \rangle_1 + \|\Delta W^j\|_1^2,$$

where $C = C(k_0)$. Summing up from $j = 0$ to $j = N - 1$, we arrive at

$$(4.9) \quad \|X^N\|_1^2 + \sum_{j=1}^N k\|X^j\|_2^2 \leq \|X_0\|_1^2 + C \sum_{j=0}^{N-1} \left(k\|X^j\|_1^2 + \langle Z^j, \Delta W^j \rangle_1 + \|\Delta W^j\|_1^2 \right).$$

Neglecting, for the moment, the sum on the left side of (4.9), raising the remaining terms to the power $p \geq 1$, and recalling (2.7), we have (with $C = C(p, k_0)$)

$$\begin{aligned} \|X^N\|_1^{2p} &\leq C \left(\|X_0\|_1^{2p} + \left(\sum_{j=0}^{N-1} k \|X^j\|_1^2 \right)^p \right. \\ &\quad \left. + \left| \sum_{j=0}^{N-1} \langle Z^j, \Delta W^j \rangle_1 \right|^p + \left(\sum_{j=0}^{N-1} \|\Delta W^j\|_1^2 \right)^p \right) \\ &\leq C \left(\|X_0\|_1^{2p} + N^{p-1} k^{p-1} \sum_{j=0}^{N-1} k \|X^j\|_1^{2p} \right. \\ &\quad \left. + \left| \sum_{j=0}^{N-1} \langle Z^j, \Delta W^j \rangle_1 \right|^{2p} + N^{p-1} \sum_{j=0}^{N-1} \|\Delta W^j\|_1^{2p} \right). \end{aligned}$$

Here we take the supremum in time and then the expectation, make repeated use of (2.7), and apply (2.5), the Burkholder–Davis–Gundy inequality (2.4) and Lemma 3.1 to conclude that, with $C = C(p, k_0, t_N, \mathbf{E}\|X_0\|_1^{2p}, \|A^{1/2}Q^{1/2}\|_{\text{HS}})$ changing from step to step,

$$\begin{aligned} \mathbf{E} \sup_{0 \leq j \leq N} \|X^j\|_1^{2p} &\leq C \left(1 + t_N^{p-1} \sum_{l=0}^{N-1} k \mathbf{E} \sup_{0 \leq j \leq l} \|X^j\|_1^{2p} \right. \\ &\quad \left. + \mathbf{E} \sup_{0 \leq n \leq N} \left| \sum_{j=0}^{n-1} \langle Z^j, \Delta W^j \rangle_1 \right|^{2p} + N^{p-1} \sum_{j=0}^{N-1} \mathbf{E} \|\Delta W^j\|_1^{2p} \right) \\ &\leq C \left(1 + t_N^{p-1} \sum_{l=0}^{N-1} k \mathbf{E} \sup_{0 \leq j \leq l} \|X^j\|_1^{2p} \right. \\ (4.10) \quad &\quad \left. + \mathbf{E} \left(\sum_{j=0}^{N-1} k \|A^{1/2}Q^{1/2}A^{1/2}Z^j\|^2 \right)^p + N^{p-1} \sum_{j=0}^{N-1} (\mathbf{E} \|\Delta W^j\|_1^2)^p \right) \\ &\leq C \left(1 + t_N^{p-1} \sum_{l=0}^{N-1} k \mathbf{E} \sup_{0 \leq j \leq l} \|X^j\|_1^{2p} \right. \\ &\quad \left. + t_N^{p-1} \sum_{j=0}^{N-1} k \mathbf{E} \|A^{1/2}Q^{1/2}\|^{2p} \|Z^j\|_1^{2p} + t_N^{p-1} \sum_{j=0}^{N-1} k \|A^{1/2}Q^{1/2}\|_{\text{HS}}^{2p} \right) \\ &\leq C \left(1 + \sum_{l=0}^{N-1} k \mathbf{E} \sup_{0 \leq j \leq l} \|X^j\|_1^{2p} \right). \end{aligned}$$

Gronwall's lemma thus yields $C = C(p, k_0, t_N, \mathbf{E}\|X_0\|_1^{2p}, \|A^{1/2}Q^{1/2}\|_{\text{HS}})$ such that

$$\mathbf{E} \sup_{0 \leq j \leq N} \|X^j\|_1^{2p} \leq C.$$

Also, since $\|J_k R_{\frac{k}{2}} X^j\|_1^2 \leq C(k_0) \|X^j\|_1^2$ by (3.24) and the contraction property of $R_{\frac{k}{2}}$ on \dot{H}^1 , it follows that

$$\mathbf{E} \sup_{0 \leq j \leq N} \|J_k R_{\frac{k}{2}} X^j\|_1^{2p} \leq C.$$

Furthermore, returning to (4.9), we easily show that, if $\mathbf{E}\|X_0\|_2^2 < \infty$, then

$$\mathbf{E} \sum_{j=0}^N k \|X^j\|_2^2 \leq C(k_0, t_N, \mathbf{E}\|X_0\|_1^2, \mathbf{E}\|X_0\|_2^2, \|A^{1/2}Q^{1/2}\|_{\text{HS}}).$$

We summarise the above findings in the following lemma.

Lemma 4.4. *If Assumption 4.1 is satisfied with $q = 2p$ for some $p \geq 1$, and $\{X^j\}_{j=0}^N$ is the solution of (1.2)–(1.5), then*

$$\mathbf{E} \sup_{0 \leq j \leq N} \|X^j\|^{2p} + \mathbf{E} \sup_{0 \leq j \leq N} \|X^j\|_1^{2p} + \mathbf{E} \sup_{0 \leq j \leq N} \|J_k R_{\frac{k}{2}} X^j\|_1^{2p} \leq C_1,$$

where $C_1 = C_1(k_0, p, T, \|A^{1/2}Q^{1/2}\|_{\text{HS}}, \mathbf{E}\|X_0\|_1^{2p})$. If, in addition, $\mathbf{E}\|X_0\|_2^2 < \infty$, then also

$$\mathbf{E} \sum_{j=0}^N k \|X^j\|_2^2 \leq C_2,$$

where $C_2 = C_2(k_0, T, \|A^{1/2}Q^{1/2}\|_{\text{HS}}, \mathbf{E}\|X_0\|_1^2, \mathbf{E}\|X_0\|_2^2)$.

4.3. Auxiliary time interpolations. For the error analysis in Section 5 we introduce two \mathcal{F}_t -measurable time interpolations of X^j : the piecewise constant function

$$(4.11) \quad \bar{X}(t) = X^j, \quad t \in [t_j, t_{j+1}),$$

and the continuous stochastic interpolation

$$(4.12) \quad \begin{aligned} \hat{X}(t) &= X^j - (t - t_j)\mathcal{A}_k X^j - (t - t_j)F_k(X^j) \\ &\quad + R_{\frac{k}{2}}(W(t) - W(t_j)), \quad t \in [t_j, t_{j+1}). \end{aligned}$$

Note that

$$\hat{X}(t) = X_0 - \int_0^t (\mathcal{A}_k \bar{X}(s) + F_k(\bar{X}(s))) \, ds + \int_0^t R_{\frac{k}{2}} \, dW(s)$$

and $\hat{X}(t_j) = \bar{X}(t_j) = X^j$.

We will also need a certain bound on the stochastic interpolation, \hat{X} . It is convenient to do this with the help of a result on the difference between \hat{X} and \bar{X} . As we will need a series of such results below we begin by proving these.

Proposition 4.5. *If Assumption 4.1 is satisfied with $q = 6p$, then there exists $C = C(k_0, p, T, \|A^{1/2}Q^{1/2}\|_{\text{HS}}, \mathbf{E}\|X_0\|_1^{6p})$ such that*

$$\sup_{0 \leq t \leq T} \mathbf{E}\|\hat{X}(t) - \bar{X}(t)\|^{2p} \leq Ck^p.$$

Proof. Note that if $t \in [t_j, t_{j+1})$, then, from the definitions of \bar{X} and \hat{X} in (4.11) and (4.12), respectively, we have that

$$\hat{X}(t) - \bar{X}(t) = -(t - t_j)\mathcal{A}_k X^j - (t - t_j)F_k(X^j) + \int_{t_j}^t R_{\frac{k}{2}} \, dW(s).$$

Therefore,

$$(4.13) \quad \begin{aligned} \sup_{t_j \leq t \leq t_{j+1}} \mathbf{E}\|\hat{X}(t) - \bar{X}(t)\|^{2p} &\leq C(p) \left(k^{2p} \mathbf{E}\|\mathcal{A}_k X^j\|^{2p} + k^{2p} \mathbf{E}\|F_k(X^j)\|^{2p} \right. \\ &\quad \left. + \sup_{t_j \leq t \leq t_{j+1}} \mathbf{E}\left\| \int_{t_j}^t R_{\frac{k}{2}} \, dW(s) \right\|^{2p} \right). \end{aligned}$$

By (3.10) and (3.14), we obtain

$$(4.14) \quad \|\mathcal{A}_k X^j\| \leq k^{-1/2} C \|X^j\|_1.$$

Moreover,

$$(4.15) \quad \begin{aligned} \|F_k(X^j)\| &= \|R_{\frac{k}{2}} f_k(R_{\frac{k}{2}} X^j)\| \leq k^{-1/2} C \|A^{-1/2} f_k(R_{\frac{k}{2}} X^j)\| \\ &\leq k^{-1/2} C(k_0) P_2(\|X^j\|_1) \|X^j\| \leq k^{-1/2} C(k_0) P_3(\|X^j\|_1), \end{aligned}$$

where we used, again, (3.14) and (1.7), utilising also that $f(0) = 0$, (3.24) and finally the boundedness of $R_{\frac{k}{2}}$. For the stochastic integral, by (2.6),

$$(4.16) \quad \sup_{t_j \leq t \leq t_{j+1}} \mathbf{E} \left\| \int_{t_j}^t R_{\frac{k}{2}} dW(s) \right\|^{2p} \leq C(p) k^p \|R_{\frac{k}{2}} Q^{1/2}\|_{\text{HS}}^{2p}.$$

Thus, inserting the bounds in (4.14), (4.15), and (4.16) into (4.13), we see that

$$\begin{aligned} \sup_{0 \leq t \leq T} \mathbf{E} \|\hat{X}(t) - \bar{X}(t)\|^{2p} &= \sup_{0 \leq j \leq N-1} \sup_{t_j \leq t \leq t_{j+1}} \mathbf{E} \|\hat{X}(t) - \bar{X}(t)\|^{2p} \\ &\leq k^p C(k_0, p, \|A^{1/2} Q^{1/2}\|_{\text{HS}}) \left(1 + \sup_{0 \leq j \leq N} \mathbf{E} P_{6p}(\|X^j\|_1) \right) \\ &\leq k^p C(k_0, p, \|A^{1/2} Q^{1/2}\|_{\text{HS}}) \left(1 + \mathbf{E} \sup_{0 \leq j \leq N} P_{6p}(\|X^j\|_1) \right). \end{aligned}$$

The claim now follows from Lemma 4.4. \square

Proposition 4.6. *Assume that Assumption 4.1 is satisfied with $q = 6$ and also $\mathbf{E}\|X_0\|_2^2 < \infty$. Then there is $C = C(k_0, T, \|A^{1/2} Q^{1/2}\|_{\text{HS}}, \mathbf{E}\|X_0\|_1^6, \mathbf{E}\|X_0\|_2^2)$ such that*

$$\int_0^t \mathbf{E} \|\hat{X}(s) - \bar{X}(s)\|_1^2 ds \leq Ck, \quad t \in [0, T].$$

Proof. As in the beginning of the previous proof we have, for $t \in [t_j, t_{j+1})$, that

$$\begin{aligned} \mathbf{E} \|\hat{X}(t) - \bar{X}(t)\|_1^2 &\leq k \left(k(\mathbf{E} \|\mathcal{A}_k X^j\|_1^2 + \mathbf{E} \|F_k(X^j)\|_1^2) + \|R_{\frac{k}{2}} Q^{1/2}\|_{\text{HS}}^2 \right) \\ &\leq k \left(\mathbf{E} \|X^j\|_2^2 + \mathbf{E} \|f_k(R_{\frac{k}{2}} X^j)\|^2 + \|R_{\frac{k}{2}} Q^{1/2}\|_{\text{HS}}^2 \right), \end{aligned}$$

as $\|\mathcal{A}_k X^j\|_1 \leq k^{-1/2} C \|X^j\|_2$ and that

$$\|F_k(X^j)\|_1 = \|A^{1/2} R_{\frac{k}{2}} f_k(R_{\frac{k}{2}} X^j)\| \leq k^{-1/2} C \|f_k(R_{\frac{k}{2}} X^j)\|.$$

Also, $\|f_k(R_{\frac{k}{2}} X^j)\| \leq C(k_0) P_3(\|X^j\|_1)$ so if $t \leq t_{n+1}$, then

$$\begin{aligned} \int_0^t \mathbf{E} \|\hat{X}(t) - \bar{X}(t)\|_1^2 &\leq C(k_0) k \sum_{j=0}^n k \left(\mathbf{E} \|X^j\|_2^2 + \mathbf{E} P_6(\|X^j\|_1) + \|R_{\frac{k}{2}} Q^{1/2}\|_{\text{HS}}^2 \right) \\ &\leq kC(k_0, T, \|A^{1/2} Q^{1/2}\|_{\text{HS}}) \left(1 + \mathbf{E} \sup_{0 \leq j \leq n} \|X^j\|_1^6 + \sum_{j=0}^n k \mathbf{E} \|X^j\|_2^2 \right). \end{aligned}$$

The claim now follows from Lemma 4.4. \square

We will use the following result.

Lemma 4.7. *If Assumption 4.1 is satisfied with $q = 6p$, then there is a constant $C = C(k_0, p, T, \|A^{1/2}Q^{1/2}\|_{\text{HS}}, \mathbf{E}\|X_0\|_1^{6p})$ such that*

$$\int_0^t \mathbf{E}\|\hat{X}(s) - \bar{X}(s)\|_1^{2p} ds \leq C, \quad t \in [0, T].$$

Proof. Using that $\|\mathcal{A}_k X^j\|_1 \leq k^{-1}C\|X^j\|_1$ and $\|F_k(X^j)\| \leq k^{-1}CP_3(\|X^j\|_1)$, it follows in a similar way as in the proof of Proposition 4.5 that if $t \leq t_{n+1}$, then

$$\int_0^t \mathbf{E}\|\hat{X}(s) - \bar{X}(s)\|_1^{2p} ds \leq C(k_0, p, T, \|A^{1/2}Q^{1/2}\|_{\text{HS}}) \left(1 + \mathbf{E} \sup_{0 \leq j \leq n} \|X^j\|_1^{6p}\right).$$

□

Corollary 4.8. *If Assumption 4.1 is satisfied with $q = 6p$, then there is $C = C(k_0, p, T, \|A^{1/2}Q^{1/2}\|_{\text{HS}}, \mathbf{E}\|X_0\|_1^{6p})$ such that*

$$\mathbf{E} \int_0^t \|\hat{X}(s)\|_1^{2p} ds \leq C, \quad t \in [0, T].$$

Proof. As $\|\hat{X}(s)\|_1 \leq \|\bar{X}(s)\|_1 + \|\bar{X}(s) - \hat{X}(s)\|_1$, the claim follows from Lemma 4.4 and Lemma 4.7. □

5. ERROR BOUNDS

We shall analyse the error $e(t)$ by splitting it as

$$e(t) := X(t) - \hat{X}(t) = (X(t) - \tilde{X}_k(t)) + (\tilde{X}_k(t) - \hat{X}(t)) =: e^1(t) + e^2(t).$$

5.1. A bound for e^1 . We start comparing the solution of the perturbed problem (3.5) to the solution of the original problem (1.1).

The main line of argument is as follows. In Lemma 5.2 we establish that \tilde{X}_k converges in $L_2(\Omega, \mathcal{F}, P; C([0, T], H))$ to a mild solution of Allen–Cahn equation (1.1), in particular, to the unique variational solution. We show, in the proof of Theorem 5.3, that the sequence $\{\tilde{X}_k\}_{k>0}$ satisfies

$$\mathbf{E} \sup_{0 \leq t \leq T} \|\tilde{X}_{k_1}(t) - \tilde{X}_{k_2}(t)\|^2 \leq Ck_2, \quad k_1 \leq k_2,$$

for some constant C depending on the initial data and Q . Thus, by taking $k_1 \rightarrow 0$ we get the desired bound for e_1 .

The first step is to analyse the error in the stochastic convolution terms. The proof is analogous to that of [22, Theorem 2.1] and is based on the factorisation method of DaPrato and Zabczyk [6, Chapter 5] and the deterministic error estimates from Lemma 3.2. We omit the details of the proof. We also note that the rate of convergence is suboptimal in terms of the regularity of the noise but a sharper (almost order k instead of $k^{1/2}$) estimate is not needed for our purposes and would require an extended range for s and r in the deterministic error estimates in Lemma 3.2 (compare with [22, (2.2)]).

Lemma 5.1. *Let $W_A(t) = \int_0^t E(t-s) dW(s)$ and $W_{\mathcal{A}_k}(t) = \int_0^t E_k(t-s) R_{\frac{k}{2}} dW(s)$. Assume $\|A^{1/2}Q^{1/2}\|_{\text{HS}}^2 < \infty$ and $2p \geq 1$. Then, with $C = C(T, p, \|A^{1/2}Q^{1/2}\|_{\text{HS}})$,*

$$(5.1) \quad \mathbf{E} \sup_{0 \leq t \leq T} \|W_A(t) - W_{\mathcal{A}_k}(t)\|^{2p} \leq Ck^p.$$

Next we prove that \tilde{X}_k converges uniformly strongly to X but with no specific rate given. The proof is in the same spirit as the proof of [21, Theorem 5.4] (see also [22]).

Lemma 5.2. *Let \tilde{X}_k be the solution of (3.5) and X of (1.1). If Assumption 4.1 is satisfied with $q = 4$, then*

$$\lim_{k \rightarrow 0} \mathbf{E} \sup_{0 \leq t \leq T} \|X(t) - \tilde{X}_k(t)\|^2 = 0.$$

Proof. It follows from Theorem 2.1, Lemma 4.2, and Lemma 5.1 that there is $K_T > 1$ such that

$$\mathbf{E} \sup_{0 \leq s \leq T} (\|X(s)\|_1^2 + \|X(s)\|^4) \leq K_T, \quad \mathbf{E} \sup_{0 \leq s \leq T} (\|\tilde{X}_k(s)\|_1^2 + \|\tilde{X}_k(s)\|^4) \leq K_T$$

and $\mathbf{E} \sup_{0 \leq t \leq T} \|W_A(t) - W_{\mathcal{A}_k}(t)\|^2 \leq K_T k$. Therefore, by Chebychev's inequality, for every $0 < \epsilon < 1$ and $0 < k < k_0$, there is $\Omega_{\epsilon, k} \subset \Omega$ with $P(\Omega_{\epsilon, k}^c) < \epsilon$ such that uniformly for all $\omega \in \Omega_{\epsilon, k}$ and $t \in [0, T]$ we have

$$(5.2) \quad \max(\|X(t)\|_1^2 + \|X(t)\|^4, \|\tilde{X}_k(t)\|_1^2 + \|\tilde{X}_k(t)\|^4) \leq K_T \epsilon^{-1},$$

$$(5.3) \quad \|W_A(t) - W_{\mathcal{A}_k}(t)\|^2 \leq K_T k \epsilon^{-1}.$$

Let $0 < \epsilon < 1$, $0 < k < k_0$ and $\omega \in \Omega_{\epsilon, k}$ and without the loss of generality suppose that also $\|X_0\|_1 \leq K_T \epsilon^{-1}$ on $\Omega_{\epsilon, k}$. Using the mild formulations (2.9) and (3.6) we see that

$$(5.4) \quad \begin{aligned} \|X(t) - \tilde{X}_k(t)\| &\leq \|(E(t) - E_k(t))X_0\| \\ &+ \left\| \int_0^t (E(t-s)f(X(s)) - E_k(t-s)F_k(\tilde{X}_k(s))) \, ds \right\| \\ &+ \|W_A(t) - W_{\mathcal{A}_k}(t)\|. \end{aligned}$$

We have, immediately from (3.18) that

$$(5.5) \quad \sup_{0 \leq t \leq T} \|(E(t) - E_k(t))X_0\| \leq k^{1/2} C \|X_0\|_1 \leq C K_T \epsilon^{-1} k^{1/2}$$

and from (5.3)

$$(5.6) \quad \sup_{0 \leq t \leq T} \|W_A(t) - W_{\mathcal{A}_k}(t)\| \leq K_T^{1/2} \epsilon^{-1/2} k^{1/2}.$$

To analyse the second term we split the integrand as

$$(5.7) \quad \begin{aligned} E(t-s)(f(X(s)) - f(\tilde{X}_k(s))) &+ (E(t-s) - E_k(t-s)R_{\frac{k}{2}})f(\tilde{X}_k(s)) \\ &+ E_k(t-s)R_{\frac{k}{2}}(f(\tilde{X}_k(s)) - f_k(R_{\frac{k}{2}}\tilde{X}_k(s))) =: e_1 + e_2 + e_3. \end{aligned}$$

Applying (2.3) and (1.7) yields

$$\begin{aligned} \|e_1\| &= \|A^{1/2}E(t-s)A^{-1/2}(f(X(s)) - f(\tilde{X}_k(s)))\| \\ &\leq C(t-s)^{-1/2} \|A^{-1/2}(f(X(s)) - f(\tilde{X}_k(s)))\| \\ &\leq (t-s)^{-1/2} (\|X(s)\|_1^2 + \|\tilde{X}_k(s)\|_1^2 + 1) \|X(s) - \tilde{X}_k(s)\|, \end{aligned}$$

so that

$$\begin{aligned}
(5.8) \quad & \left\| \int_0^t e_1 \, ds \right\| \leq C \sup_{0 \leq s \leq T} (\|X(s)\|_1^2 + \|\tilde{X}_k(s)\|_1^2 + 1) \\
& \times \int_0^t (t-s)^{-1/2} \|X(s) - \tilde{X}_k(s)\| \, ds \\
& \leq C(1 + 2K_T \epsilon^{-1}) \int_0^t (t-s)^{-1/2} \|X(s) - \tilde{X}_k(s)\| \, ds.
\end{aligned}$$

By Lemma 3.2 we have that

$$\|e_2\| \leq C(t-s)^{-1/2} \|f(\tilde{X}_k(s))\| k^{1/2},$$

and from (5.2) it follows that

$$\|f(\tilde{X}_k(s))\| \leq C(\|\tilde{X}_k(s)\|_1^3 + 1) \leq C(1 + K_T^{3/2} \epsilon^{-3/2}), \quad s \in [0, T].$$

Thus,

$$(5.9) \quad \sup_{0 \leq t \leq T} \left\| \int_0^t e_2(s) \, ds \right\| \leq C(1 + K_T^{3/2} \epsilon^{-3/2}) k^{1/2}.$$

We continue with the observation that

$$\begin{aligned}
e_3 &= (AM_k)^{1/2} R_{\frac{k}{2}} E_k(t-s) (AM_k)^{-1/2} (f(\tilde{X}_k(s)) - f(J_k R_{\frac{k}{2}} \tilde{X}_k(s))) \\
&= \mathcal{A}_k^{1/2} E_k(t-s) (AM_k)^{-1/2} (f(R_{\frac{k}{2}} \tilde{X}_k(s)) - f(\tilde{X}_k(s))).
\end{aligned}$$

Therefore, by (2.3), (1.7), and (3.34),

$$\begin{aligned}
\|e_3\| &\leq (t-s)^{-1/2} C \|(AM_k)^{-1/2} (f(R_{\frac{k}{2}} \tilde{X}_k(s)) - f(\tilde{X}_k(s)))\| \\
&\leq (t-s)^{-1/2} C \|A^{-1/2} (f(R_{\frac{k}{2}} \tilde{X}_k(s)) - f(\tilde{X}_k(s)))\| \\
&\leq (t-s)^{-1/2} C (\|\tilde{X}_k(s)\|_1^2 + 1) \|\tilde{X}_k(s) - J_k R_{\frac{k}{2}} \tilde{X}_k(s)\| \\
&\leq (t-s)^{-1/2} C (\|\tilde{X}_k(s)\|_1^2 + 1) k (\|f(J_k R_{\frac{k}{2}} \tilde{X}_k(s))\| + \|AR_{\frac{k}{2}} \tilde{X}_k(s)\|).
\end{aligned}$$

As $\|AR_{\frac{k}{2}} \tilde{X}_k(s)\| \leq Ck^{-1/2} \|\tilde{X}_k(s)\|_1$ and $\|f(J_k R_{\frac{k}{2}} \tilde{X}_k(s))\| \leq C(\|\tilde{X}_k(s)\|_1^3 + 1)$, we get in the same way as above that

$$(5.10) \quad \sup_{0 \leq t \leq T} \left\| \int_0^t e_3(s) \, ds \right\| \leq C(1 + K_T \epsilon^{-1}) K_T^{1/2} \epsilon^{-1/2} C(1 + K_T^{3/2} \epsilon^{-3/2}) k^{1/2}.$$

Summarising the above findings, we have

$$\begin{aligned}
\|X(t) - \tilde{X}_k(t)\| &\leq C(1 + K_T^3 \epsilon^{-3}) k^{1/2} \\
&\quad + C(1 + 2K_T \epsilon^{-1}) \int_0^t (t-s)^{-1/2} \|X(s) - \tilde{X}_k(s)\| \, ds.
\end{aligned}$$

It follows from Gronwall's lemma, Lemma 2.2, that there is $C_\epsilon = C(T, K_T \epsilon^{-1})$ (growing rapidly as $\epsilon \rightarrow 0$) such that uniformly on $\Omega_{\epsilon, k}$,

$$(5.11) \quad \sup_{0 \leq t \leq T} \|X(t) - \tilde{X}_k(t)\| \leq C_\epsilon k^{1/2}.$$

Despite the large constant in (5.11), it is now possible to prove strong convergence of \tilde{X}_k to X but with no rate. Indeed,

$$\begin{aligned}
\mathbf{E} \sup_{0 \leq t \leq T} \|X(t) - \tilde{X}_k(t)\|^2 &\leq \int_{\Omega_{\epsilon,k}} \sup_{0 \leq t \leq T} \|X(t) - \tilde{X}_k(t)\|^2 dP \\
&\quad + 2 \int_{\Omega_{\epsilon,k}^c} \sup_{0 \leq t \leq T} \left(\|X(t)\|^2 + \|\tilde{X}_k(t)\|^2 \right) dP \\
&\leq C_\epsilon^2 k + 4\epsilon^{1/2} \left(\int_{\Omega_{\epsilon,k}^c} \sup_{0 \leq t \leq T} \left(\|X(t)\|^4 + \|\tilde{X}_k(t)\|^4 \right) dP \right)^{1/2} \\
&\leq C_\epsilon^2 k + 4\epsilon^{1/2} \left(\mathbf{E} \sup_{0 \leq t \leq T} \left(\|X(t)\|^4 + \|\tilde{X}_k(t)\|^4 \right) \right)^{1/2} \\
&\leq C_\epsilon^2 k + 8\epsilon^{1/2} K_T^{1/2}.
\end{aligned}$$

Let $\delta > 0$ and choose $0 < \epsilon < 1$ such that $8\epsilon^{1/2} K_T^{1/2} < \delta/2$. Then, for $k < \frac{1}{2} C_\epsilon^{-2} \delta$, we have that $\mathbf{E} \sup_{0 \leq t \leq T} \|X(t) - \tilde{X}_k(t)\|^2 < \delta$ and the proof is complete. \square

Now, we are ready to prove the error estimate. The proof of the fact that $\{\tilde{X}_k\}_{k>0}$ is a Cauchy sequence benefits from techniques in [5, Chapter 4].

Theorem 5.3. *Let \tilde{X}_k be the solution of (3.5) and X of (1.1). If Assumption 4.1 is satisfied with $q = 6$, then there is $C = C(k_0, T, \|A^{1/2}Q^{1/2}\|_{\text{HS}}, \mathbf{E}\|X_0\|_1^6)$ such that*

$$(5.12) \quad \mathbf{E} \sup_{0 \leq t \leq T} \|\tilde{X}_k(t) - X(t)\|^2 \leq Ck.$$

We suppress the dependency of the constants C on the data in order to, we hope, increase the readability of the following proof.

Proof. We start by proving that $\{\tilde{X}_k\}_{k>0}$ is Cauchy in $L_2(\Omega, \mathcal{F}, P; C([0, T], H))$. To this aim we pick two time-steps δ and γ and assume, without loss of generality, that $\gamma \leq \delta \leq k_0$. The corresponding solutions of (3.5) are \tilde{X}_δ and \tilde{X}_γ . We note that $\tilde{X}_\delta(t) - \tilde{X}_\gamma(t)$ solves the equation

$$\begin{aligned}
&d(\tilde{X}_\delta - \tilde{X}_\gamma) + (\mathcal{A}_\delta \tilde{X}_\delta - \mathcal{A}_\gamma \tilde{X}_\gamma) dt + (F_\delta(\tilde{X}_\delta) - F_\gamma(\tilde{X}_\gamma)) dt \\
&= (R_{\frac{\delta}{2}} - R_{\frac{\gamma}{2}}) dW, \quad t \in (0, T]; \quad \tilde{X}_\delta(0) - \tilde{X}_\gamma(0) = 0.
\end{aligned}$$

Applying Itô's formula to $\frac{1}{2} \|\tilde{X}_\delta(t) - \tilde{X}_\gamma(t)\|^2$ thus gives

$$\begin{aligned}
\frac{1}{2} \|\tilde{X}_\delta(t) - \tilde{X}_\gamma(t)\|^2 &= - \int_0^t \langle \tilde{X}_\delta(s) - \tilde{X}_\gamma(s), \mathcal{A}_\delta \tilde{X}_\delta(s) - \mathcal{A}_\gamma \tilde{X}_\gamma(s) \rangle ds \\
&\quad - \int_0^t \langle \tilde{X}_\delta(s) - \tilde{X}_\gamma(s), F_\delta(\tilde{X}_\delta(s)) - F_\gamma(\tilde{X}_\gamma(s)) \rangle ds \\
&\quad + \int_0^t \langle \tilde{X}_\delta(s) - \tilde{X}_\gamma(s), (R_{\frac{\delta}{2}} - R_{\frac{\gamma}{2}}) dW(s) \rangle \\
&\quad + \frac{1}{2} \int_0^t \text{Tr}((R_{\frac{\delta}{2}} - R_{\frac{\gamma}{2}}) Q^{1/2} ((R_{\frac{\delta}{2}} - R_{\frac{\gamma}{2}}) Q^{1/2})^*) ds \\
&=: -I_1 - I_2 + I_3 + I_4.
\end{aligned}$$

For I_4 we note that, by (3.15) and (3.16),

$$\begin{aligned} \text{Tr}((R_{\frac{\delta}{2}} - R_{\frac{\gamma}{2}})Q^{1/2}((R_{\frac{\delta}{2}} - R_{\frac{\gamma}{2}})Q^{1/2})^*) &= \|(R_{\frac{\delta}{2}} - R_{\frac{\gamma}{2}})Q^{1/2}\|_{\text{HS}}^2 \\ &\leq C\delta^2 \|AR_{\frac{\delta}{2}}R_{\frac{\gamma}{2}}Q^{1/2}\|_{\text{HS}}^2 \leq C\delta \|A^{1/2}R_{\frac{\gamma}{2}}Q^{1/2}\|_{\text{HS}}^2. \end{aligned}$$

Thus,

$$(5.13) \quad \mathbf{E} \sup_{0 \leq t \leq T} |I_4(t)| \leq \delta CT \|A^{1/2}Q^{1/2}\|_{\text{HS}}^2.$$

For I_3 we have, using again (3.15), that

$$\langle \tilde{X}_\delta(s) - \tilde{X}_\gamma(s), (R_{\frac{\delta}{2}} - R_{\frac{\gamma}{2}}) dW(s) \rangle = \frac{1}{2}(\delta - \gamma) \langle \tilde{X}_\delta(s) - \tilde{X}_\gamma(s), AR_{\frac{\delta}{2}}R_{\frac{\gamma}{2}} dW(s) \rangle.$$

Hence, using Cauchy's inequality, the Burkholder–Davies–Gundy inequality (2.4), Hölder's inequality, (3.14), and Lemma 3.1, we have

$$\begin{aligned} (5.14) \quad \mathbf{E} \sup_{0 \leq t \leq T} |I_3(t)| &\leq C\delta \mathbf{E} \sup_{0 \leq t \leq T} \left| \int_0^t \langle \tilde{X}_\delta(s) - \tilde{X}_\gamma(s), AR_{\frac{\delta}{2}}R_{\frac{\gamma}{2}} dW(s) \rangle \right| \\ &\leq C\delta \left(1 + \mathbf{E} \sup_{0 \leq t \leq T} \left| \int_0^t \langle \tilde{X}_\delta(s) - \tilde{X}_\gamma(s), AR_{\frac{\delta}{2}}R_{\frac{\gamma}{2}} dW(s) \rangle \right|^2 \right) \\ &\leq C\delta \left(1 + \mathbf{E} \int_0^T \|Q^{1/2}A^{1/2}R_{\frac{\gamma}{2}}A^{1/2}R_{\frac{\delta}{2}}(\tilde{X}_\delta(s) - \tilde{X}_\gamma(s))\|^2 ds \right) \\ &\leq C\delta \left(1 + \mathbf{E} \int_0^T \|Q^{1/2}A^{1/2}R_{\frac{\gamma}{2}}\|^2 \|A^{1/2}R_{\frac{\delta}{2}}(\tilde{X}_\delta(s) - \tilde{X}_\gamma(s))\|^2 ds \right) \\ &\leq C\delta \left(1 + \frac{1}{\delta} \mathbf{E} \int_0^T \|Q^{1/2}A^{1/2}R_{\frac{\gamma}{2}}\|^2 \|\tilde{X}_\delta(s) - \tilde{X}_\gamma(s)\|^2 ds \right) \\ &\leq C \left(\delta + \int_0^T \mathbf{E} \sup_{0 \leq t \leq s} \|\tilde{X}_\delta(t) - \tilde{X}_\gamma(t)\|^2 ds \right). \end{aligned}$$

To squeeze out smallness of I_2 we first note that by (1.6),

$$\begin{aligned} a_2^+ &:= \langle f(J_\delta R_{\frac{\delta}{2}} \tilde{X}_\delta(s)) - f(J_\gamma R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s)), J_\delta R_{\frac{\delta}{2}} \tilde{X}_\delta(s) - J_\gamma R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s) \rangle \\ &\quad + \beta^2 \|J_\delta R_{\frac{\delta}{2}} \tilde{X}_\delta(s) - J_\gamma R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s)\|^2 \geq 0. \end{aligned}$$

From (3.25) it follows that

$$\begin{aligned} \|J_\delta R_{\frac{\delta}{2}} \tilde{X}_\delta(s) - J_\gamma R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s)\|^2 &\leq C\delta^2 \left(\|f_\delta(R_{\frac{\delta}{2}} \tilde{X}_\delta(s))\|^2 + \|f_\gamma(R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s))\|^2 \right) \\ &\quad + C \|R_{\frac{\delta}{2}} \tilde{X}_\delta(s) - R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s)\|^2, \end{aligned}$$

and from (3.15) that

$$\begin{aligned} \|R_{\frac{\delta}{2}} \tilde{X}_\delta(s) - R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s)\| &\leq \|R_{\frac{\delta}{2}} \tilde{X}_\delta(s) - R_{\frac{\delta}{2}} \tilde{X}_\gamma(s)\| + \|R_{\frac{\delta}{2}} \tilde{X}_\gamma(s) - R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s)\| \\ &\leq C \left(\|\tilde{X}_\delta(s) - \tilde{X}_\gamma(s)\| + \delta \|AR_{\frac{\delta}{2}}R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s)\| \right). \end{aligned}$$

Combining the last three inequalities, inserting $f_\alpha = f(J_\alpha \cdot)$, we find that

$$\begin{aligned} (5.15) \quad 0 \leq a_2^+ &\leq \langle f_\delta(R_{\frac{\delta}{2}} \tilde{X}_\delta(s)) - f_\gamma(R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s)), J_\delta R_{\frac{\delta}{2}} \tilde{X}_\delta(s) - J_\gamma R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s) \rangle \\ &\quad + C \left(\|\tilde{X}_\delta(s) - \tilde{X}_\gamma(s)\|^2 + \delta^2 (\|f_\delta(R_{\frac{\delta}{2}} \tilde{X}_\delta(s))\|^2 \right. \\ &\quad \left. + \|f_\gamma(R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s))\|^2 + \|AR_{\frac{\delta}{2}}R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s)\|^2) \right). \end{aligned}$$

We shall subtract I_2 from $I_2^+ := \int_0^t a_2^+ ds$ but first we note, using (3.26) and (3.16) as well as Hölder's and Cauchy's inequalities, that

$$(5.16) \quad \begin{aligned} -I_2 \leq & \int_0^t \left\{ \langle f_\delta(R_{\frac{\delta}{2}} \tilde{X}_\delta(s)) - f_\gamma(R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s)), R_{\frac{\delta}{2}} \tilde{X}_\delta(s) - R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s) \rangle \right. \\ & + \delta \left(\|f_\delta(R_{\frac{\delta}{2}} \tilde{X}_\delta(s))\|^2 + \|f_\gamma(R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s))\|^2 \right. \\ & \left. \left. + \|AR_{\frac{\delta}{2}} R_{\frac{\gamma}{2}} \tilde{X}_\delta(s)\|^2 + \|AR_{\frac{\delta}{2}} R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s)\|^2 \right) \right\} ds. \end{aligned}$$

Thus, from (5.15) and (5.16), we have

$$\begin{aligned} I_2^+ - I_2 \leq & \int_0^t \left\{ \langle f_\delta(R_{\frac{\delta}{2}} \tilde{X}_\delta(s)) - f_\gamma(R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s)), J_\delta R_{\frac{\delta}{2}} \tilde{X}_\delta(s) - J_\gamma R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s) \rangle \right. \\ & - \langle f_\delta(R_{\frac{\delta}{2}} \tilde{X}_\delta(s)) - f_\gamma(R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s)), R_{\frac{\delta}{2}} \tilde{X}_\delta(s) - R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s) \rangle \\ & + C \left(\|\tilde{X}_\delta(s) - \tilde{X}_\gamma(s)\|^2 + \delta (\|f_\delta(R_{\frac{\delta}{2}} \tilde{X}_\delta(s))\|^2 + \|f_\gamma(R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s))\|^2 \right. \\ & \left. \left. + \|AR_{\frac{\delta}{2}} R_{\frac{\gamma}{2}} \tilde{X}_\delta(s)\|^2 + \|AR_{\frac{\delta}{2}} R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s)\|^2 \right) \right\} ds. \end{aligned}$$

Using (3.25) again, it follows that

$$\begin{aligned} & \langle f_\delta(R_{\frac{\delta}{2}} \tilde{X}_\delta(s)) - f_\gamma(R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s)), J_\delta R_{\frac{\delta}{2}} \tilde{X}_\delta(s) - J_\gamma R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s) \rangle \\ & - \langle f_\delta(R_{\frac{\delta}{2}} \tilde{X}_\delta(s)) - f_\gamma(R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s)), R_{\frac{\delta}{2}} \tilde{X}_\delta(s) - R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s) \rangle \\ & = -\langle f_\delta(R_{\frac{\delta}{2}} \tilde{X}_\delta(s)) - f_\gamma(R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s)), \delta f(J_\delta R_{\frac{\delta}{2}} \tilde{X}_\delta(s)) - \gamma f(J_\gamma R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s)) \rangle \\ & \leq C\delta \left(\|f_\delta(R_{\frac{\delta}{2}} \tilde{X}_\delta(s))\|^2 + \|f_\gamma(R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s))\|^2 \right). \end{aligned}$$

Inserting this into the previous inequality, taking the supremum in time and then the expectation, after some trivial estimates we may conclude that

$$(5.17) \quad \begin{aligned} \mathbf{E} \sup_{0 \leq t \leq T} |I_2^+ - I_2| \leq & C \left(\int_0^T \mathbf{E} \sup_{0 \leq s \leq t} \|\tilde{X}_\delta(s) - \tilde{X}_\gamma(s)\|^2 dt \right. \\ & + \delta \int_0^T \mathbf{E} (\|f_\delta(R_{\frac{\delta}{2}} \tilde{X}_\delta(s))\|^2 + \|f_\gamma(R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s))\|^2 \\ & \left. + \|AR_{\frac{\delta}{2}} R_{\frac{\gamma}{2}} \tilde{X}_\delta(s)\|^2 + \|AR_{\frac{\delta}{2}} R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s)\|^2) ds \right). \end{aligned}$$

The term I_1 can be dealt with in a similar fashion. Indeed, we have that

$$\begin{aligned} I_1^+ &:= \int_0^t \langle M_\delta R_{\frac{\delta}{2}}^2 \tilde{X}_\delta(s) - M_\gamma R_{\frac{\gamma}{2}}^2 \tilde{X}_\gamma(s), \mathcal{A}_\delta \tilde{X}_\delta(s) - \mathcal{A}_\gamma \tilde{X}_\gamma(s) \rangle ds \\ &= \int_0^t \|M_\delta R_{\frac{\delta}{2}}^2 \tilde{X}_\delta(s) - M_\gamma R_{\frac{\gamma}{2}}^2 \tilde{X}_\gamma(s)\|_1^2 ds \geq 0. \end{aligned}$$

Also, note that $I - \tilde{M}_k R_{\frac{k}{2}}^2 = \frac{k}{4} A(3 + kA) R_{\frac{k}{2}}^2$. Thus, abbreviating $\tilde{M}_k = (3 + kA)$, we have

$$\begin{aligned} I_1^+ - I_1 &= \frac{1}{4} \int_0^t \langle A(\delta \tilde{M}_\delta R_{\frac{\delta}{2}}^2 \tilde{X}_\delta(s) - \gamma \tilde{M}_\gamma R_{\frac{\gamma}{2}}^2 \tilde{X}_\gamma(s)), \mathcal{A}_\delta \tilde{X}_\delta(s) - \mathcal{A}_\gamma \tilde{X}_\gamma(s) \rangle ds \\ &\leq C \int_0^t (\|\mathcal{A}_\delta \tilde{X}_\delta(s)\| + \|\mathcal{A}_\gamma \tilde{X}_\gamma(s)\|) (\delta \|A(\tilde{M}_\delta R_{\frac{\delta}{2}}^2 \tilde{X}_\delta(s))\| \\ &\quad + \gamma \|A\tilde{M}_\gamma R_{\frac{\gamma}{2}}^2 \tilde{X}_\gamma(s)\|) ds. \end{aligned}$$

But $\|\tilde{M}_k x\| \leq 4\|M_k x\|$ and hence

$$(5.18) \quad \mathbf{E} \sup_{0 \leq t \leq T} |I_1^+ - I_1| \leq C\delta \int_0^T \mathbf{E} \left(\|\mathcal{A}_\delta \tilde{X}_\delta(s)\|^2 + \|\mathcal{A}_\gamma \tilde{X}_\gamma(s)\|^2 \right) ds.$$

Combining the results in (5.13), (5.14), (5.17), and (5.18), we thus conclude that

$$\begin{aligned} \mathbf{E} \sup_{0 \leq t \leq T} \frac{1}{2} \|\tilde{X}_\delta(t) - \tilde{X}_\gamma(t)\|^2 &= \mathbf{E} \sup_{0 \leq t \leq T} (-I_1 - I_2 + I_3 + I_4) \\ &\leq \mathbf{E} \sup_{0 \leq t \leq T} (I_1^+ - I_1 + I_2^+ - I_2 + I_3 + I_4) \\ &\leq C \left\{ \int_0^T \mathbf{E} \sup_{0 \leq t \leq s} \|\tilde{X}_\delta(t) - \tilde{X}_\gamma(t)\|^2 ds \right. \\ &\quad + \delta \left(1 + \int_0^T \|f_\delta(R_{\frac{\delta}{2}} \tilde{X}_\delta(s))\|^2 + \|f_\gamma(R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s))\|^2 \right. \\ &\quad + \|AR_{\frac{\delta}{2}} R_{\frac{\gamma}{2}} \tilde{X}_\delta(s)\|^2 + \|AR_{\frac{\delta}{2}} R_{\frac{\gamma}{2}} \tilde{X}_\gamma(s)\|^2 \\ &\quad \left. \left. + \|\mathcal{A}_\delta \tilde{X}_\delta(s)\|^2 + \|\mathcal{A}_\gamma \tilde{X}_\gamma(s)\|^2 ds \right) \right\}. \end{aligned}$$

The second integral is bounded by Lemma 4.2 and Corollary 4.3 since we assume $\mathbf{E}\|X_0\|_1^6 < \infty$. Note that Lemma 4.2 is applicable as, by (3.8) and (3.9), we have $\|AR_{\frac{\delta}{2}} R_{\frac{\gamma}{2}} x\| \leq 2\|\mathcal{A}_\delta x\|$ and $\|\mathcal{A}_\delta x\| \leq \|\mathcal{A}_\delta^{1/2} x\|_1$ (similarly for the terms with \mathcal{A}_γ). Thus, by Gronwall's lemma,

$$(5.19) \quad \mathbf{E} \sup_{0 \leq t \leq T} \frac{1}{2} \|\tilde{X}_\delta(t) - \tilde{X}_\gamma(t)\|^2 \leq C(T)\delta.$$

Therefore, it follows that there is \tilde{X} such that $\tilde{X}_k \rightarrow \tilde{X}$ in $L_2(\Omega, \mathcal{F}, P; C([0, T], H))$ as $k \rightarrow 0$. But according to Lemma 5.2 \tilde{X} must be X . Thus the theorem follows from (5.19) by taking $\gamma \rightarrow 0$. \square

5.2. A bound for e^2 . We have the following result.

Theorem 5.4. *If Assumption 4.1 is satisfied with $q = 18$ and $\mathbf{E}\|X_0\|_2^2 < \infty$, then there is $C = C(k_0, T, \|A^{1/2}Q^{1/2}\|_{\text{HS}}, \mathbf{E}\|X_0\|_1^{18}, \mathbf{E}\|X_0\|_2^2) > 0$ such that*

$$\mathbf{E} \sup_{0 \leq t \leq T} \|\tilde{X}_k(t) - \hat{X}(t)\|^2 \leq Ck.$$

Proof. First note that

$$d(\tilde{X}_k(t) - \hat{X}(t)) = -(\mathcal{A}_k(\tilde{X}_k(t) - \bar{X}(t)) + (F_k(\tilde{X}_k(t)) - F_k(\bar{X}(t)))) dt.$$

Thus, since $\tilde{X}_k(0) = \bar{X}(0)$, it follows that

$$(5.20) \quad \begin{aligned} \frac{1}{2} \|\tilde{X}_k(t) - \hat{X}(t)\|^2 &= - \int_0^t \langle \tilde{X}_k(s) - \hat{X}(s), \mathcal{A}_k(\tilde{X}_k(s) - \bar{X}(s)) \rangle ds \\ &\quad - \int_0^t \langle \tilde{X}_k(s) - \hat{X}(s), F_k(\tilde{X}_k(s)) - F_k(\bar{X}(s)) \rangle ds. \end{aligned}$$

We have that, for any $\epsilon_1 > 0$ there is C_1 such that

$$\begin{aligned}
& -\langle \tilde{X}_k(s) - \hat{X}(s), \mathcal{A}_k(\tilde{X}_k(s) - \bar{X}(s)) \rangle \\
& = -\langle \tilde{X}_k(s) - \hat{X}(s), \mathcal{A}_k(\tilde{X}_k(s) - \hat{X}(s)) \rangle - \langle \tilde{X}_k(s) - \hat{X}(s), \mathcal{A}_k(\hat{X}(s) - \bar{X}(s)) \rangle \\
& \leq -\|M_k^{1/2} R_{\frac{k}{2}}(\tilde{X}_k(s) - \hat{X}(s))\|_1^2 + \epsilon_1 \|M_k^{1/2} R_{\frac{k}{2}}(\tilde{X}_k(s) - \hat{X}(s))\|_1^2 \\
& \quad + C_1 \|A^{1/2} M_k^{1/2} R_{\frac{k}{2}}(\bar{X}(s) - \hat{X}(s))\|^2.
\end{aligned}$$

Similarly, using (1.6), (1.7), (3.23) and (3.7) we compute

$$\begin{aligned}
& -\langle \tilde{X}_k(s) - \hat{X}(s), F_k(\tilde{X}_k(s)) - F_k(\bar{X}(s)) \rangle \\
& = -\langle \tilde{X}_k(s) - \hat{X}(s), F_k(\tilde{X}_k(s)) - F_k(\hat{X}(s)) \rangle \\
& \quad - \langle \tilde{X}_k(s) - \hat{X}(s), F_k(\hat{X}(s)) - F_k(\bar{X}(s)) \rangle \\
& \leq C \|R_{\frac{k}{2}}(\tilde{X}_k(s) - \hat{X}(s))\|^2 \\
& \quad - \langle A^{1/2} R_{\frac{k}{2}}(\tilde{X}_k(s) - \hat{X}(s)), A^{-1/2}(f(J_k R_{\frac{k}{2}} \hat{X}(s)) - f(J_k R_{\frac{k}{2}} \bar{X}(s))) \rangle \\
& \leq C \|R_{\frac{k}{2}}(\tilde{X}_k(s) - \hat{X}(s))\|^2 + \epsilon_2 \|A^{1/2} R_{\frac{k}{2}}(\tilde{X}_k(s) - \hat{X}(s))\|^2 \\
& \quad + C_2 \|A^{-1/2} f(J_k R_{\frac{k}{2}} \hat{X}(s)) - f(J_k R_{\frac{k}{2}} \bar{X}(s))\|^2 \\
& \leq C \|R_{\frac{k}{2}}(\tilde{X}_k(s) - \hat{X}(s))\|^2 + \epsilon_2 \|R_{\frac{k}{2}}(\tilde{X}_k(s) - \hat{X}(s))\|_1^2 \\
& \quad + C_3 (\|J_k R_{\frac{k}{2}} \hat{X}(s)\|_1^4 + \|J_k R_{\frac{k}{2}} \bar{X}(s)\|_1^4 + 1) \|J_k R_{\frac{k}{2}} \hat{X}(s) - J_k R_{\frac{k}{2}} \bar{X}(s)\|^2 \\
& \leq C \|\tilde{X}_k(s) - \hat{X}(s)\|^2 + \epsilon_2 \|M^{1/2} R_{\frac{k}{2}}(\tilde{X}_k(s) - \hat{X}(s))\|_1^2 \\
& \quad + C_3 (\|J_k R_{\frac{k}{2}} \hat{X}(s)\|_1^4 + \|J_k R_{\frac{k}{2}} \bar{X}(s)\|_1^4 + 1) \|\hat{X}(s) - \bar{X}(s)\|^2.
\end{aligned}$$

Taking $\epsilon_1 + \epsilon_2 = 1$ and inserting these estimates into (5.20) we get, after taking first the supremum in time and then the expectation, that

$$\begin{aligned}
(5.21) \quad & \mathbf{E} \sup_{0 \leq t \leq T} \|\tilde{X}_k(t) - \hat{X}(t)\|^2 \leq C \left(\int_0^T \mathbf{E} \sup_{0 \leq s \leq t} \|\tilde{X}_k(s) - \hat{X}(s)\|^2 dt \right. \\
& + \mathbf{E} \int_0^T \|A^{1/2} M_k^{1/2} R_{\frac{k}{2}}(\bar{X}(s) - \hat{X}(s))\|^2 ds \\
& \left. + \mathbf{E} \int_0^T (\|J_k R_{\frac{k}{2}} \hat{X}(s)\|_1^4 + \|J_k R_{\frac{k}{2}} \bar{X}(s)\|_1^4 + 1) \|\hat{X}(s) - \bar{X}(s)\|^2 ds \right).
\end{aligned}$$

By Hölder's inequality with exponents $3/2$ and 3 , the linear growth bound (3.24) of J_k and the contractivity of $R_{\frac{k}{2}}$ in \dot{H}^1 , the last term is bounded by

$$\begin{aligned}
& C \left(\mathbf{E} \int_0^T (\|\hat{X}(s)\|_1^6 + \|\bar{X}(s)\|_1^6 + 1) ds \right)^{2/3} \left(\mathbf{E} \int_0^T (\|\hat{X}(s) - \bar{X}(s)\|^6) ds \right)^{1/3} \\
& \leq C(\mathbf{E}\|X_0\|_1^{18}, \mathbf{E}\|X_0\|_2^2, T, k_0, \|A^{1/2} Q^{1/2}\|_{\text{HS}}) k.
\end{aligned}$$

The last inequality follows since the first integral is bounded using Lemma 4.4 and Corollary 4.8 and the second integral is bounded by Ck using Proposition 4.5. As $M_k^{1/2} R_{\frac{k}{2}}$ is a bounded operator (uniformly in k) that commutes with $A^{1/2}$, Proposition 4.6 asserts the same bound on the second term in the right hand side

of (5.21) and thus

$$(5.22) \quad \mathbf{E} \sup_{0 \leq t \leq T} \|\tilde{X}_k(t) - \hat{X}(t)\|^2 \leq C \left(k + \int_0^T \mathbf{E} \sup_{0 \leq s \leq t} \|\tilde{X}_k(s) - \hat{X}(s)\|^2 dt \right),$$

and the statement of the theorem follows by Gronwall's lemma. \square

Remark 5.5. *Due to the fact that we have assumed additive noise, the noise term vanishes upon taking the difference between \hat{X} and \tilde{X} . Thus Itô's formula is not needed in the proof of Theorem 5.4. However, if multiplicative noise would have been considered and evaluated explicitly, the use of the fundamental theorem of calculus in (5.20) could be replaced by the use of Itô's formula since the corresponding construction of \hat{X} will be adapted and continuous, and, under suitable assumptions on the covariance operator, similar results may be achieved. For the BE scheme in (1.11), however, a continuous adapted interpolation cannot be constructed. See [17] for the finite dimensional case.*

5.3. Convergence of the backward Euler scheme. Finally, we show that the backward Euler scheme, (1.11), for (1.1) is an $\mathcal{O}(k^{1/2})$ perturbation of the split-step scheme (1.2)–(1.5), thus converges with the same rate to the solution of the stochastic Allen–Cahn equation.

First note that the elliptic equation

$$(5.23) \quad x + k(Ax + f(x)) = y$$

has a unique weak solution $x \in \dot{H}^1$ for every $y \in \dot{H}^1$ if $2k\beta^2 < 1$. To see this observe first that for any $K \in \mathbb{R}$ the function

$$(5.24) \quad L(\nabla u, u, \xi) = \frac{1}{2}k|\nabla u|^2 + \frac{1}{2}(1 - k\beta^2)|u|^2 + \frac{1}{4}ku^4 - uy(\xi) + Ky(\xi)^2$$

is a Lagrangian for (5.23) and if K is large enough, then it even satisfies the coercivity condition $L(\eta, \nu, \xi) \geq C|\eta|^2$. Following [8], Chapter 8.2, Theorems 1,2 and 4 the functional

$$I(v) = \int_{\mathcal{D}} L(\nabla v, v, \xi) d\xi$$

has a minimiser $u \in W_0^{1,4}$ which is a weak solution of (5.23) with test functions $v \in W_0^{1,4}$. Then $u \in \dot{H}^1$ and using first test functions $v \in C_c^\infty$ in the weak form of (5.23) and a standard approximation argument one easily sees that u is a weak solution with test functions $v \in \dot{H}^1$. We note that Theorems 1 and 2 in [8] assumes smoothness of L , which does not hold in our case in the variable ξ as y only belongs to \dot{H}^1 . However given the simple and explicit dependence of L on ξ one verifies that the proofs of Theorems 1 and 2 hold verbatim (in places even using simpler arguments) for L given by (5.24). Finally, uniqueness of weak solutions (5.23) follows as in the uniqueness part of the proof of Lemma 3.3 from (1.6) using also the positivity of A .

Hence, if X_0 and ΔW^{j-1} belongs to \dot{H}^1 almost surely, the BE scheme has a unique pathwise solution almost surely.

We need two results on Hölder regularity of X^j .

Lemma 5.6. *If, for some $p \geq 1$, Assumption 4.1 is satisfied with $q = 6p$, then there exists $C = C(k_0, p, T, \|A^{1/2}Q^{1/2}\|_{\text{HS}}, \mathbf{E}\|X_0\|_1^{6p})$ such that*

$$\sup_{1 \leq j \leq N} \mathbf{E}\|X^j - X^{j-1}\|^{2p} \leq Ck^p.$$

Lemma 5.7. *If Assumption 4.1 is satisfied with $q = 6$ and $\mathbf{E}\|X_0\|_2^2 < \infty$, then there is $C = C(k_0, T, \|A^{1/2}Q^{1/2}\|_{\text{HS}}, \mathbf{E}\|X_0\|_1^6, \mathbf{E}\|X_0\|_2^2)$ such that*

$$\sum_{j=1}^N k\mathbf{E}\|X^j - X^{j-1}\|_1^2 \leq Ck.$$

The proofs of Lemmata 5.6 and 5.7 are omitted as they are analogous to the proofs of Propositions 4.5 and 4.6, respectively, noting that $X^j - X^{j-1} = -k\mathcal{A}_k X^j - kF_k(X^j) + \int_{t_{j-1}}^{t_j} R_{\frac{k}{2}} dW(s)$.

In the error analysis below we will restrict ourselves to the discrete grid for brevity. It is worth noting that the proof of the following theorem is the fully discrete analog of the proof of Theorem 5.4.

Theorem 5.8. *Let X_{be}^j and X^j be the solution of (1.11) and the system (1.2)–(1.5), respectively. Under the same assumptions as in Theorem 5.4 there exists $C > 0$, depending on the same data as the constant in Theorem 5.4, such that*

$$(5.25) \quad \mathbf{E} \sup_{0 \leq j \leq N} \|X_{\text{be}}^j - X^j\|^2 \leq Ck.$$

Proof. We first note that the split-step scheme may be written as

$$(5.26) \quad \begin{aligned} & X^j - X^{j-1} + kAX^j \\ & - \frac{k}{2}A(X^j - X^{j-1} + \frac{k}{2}AR_{\frac{k}{2}}X^{j-1}) + kf_k(R_{\frac{k}{2}}X^{j-1}) = \Delta W^{j-1}, \end{aligned}$$

for $j = 1, 2, \dots, N$ with $X^0 = X_0$. This is easiest seen by multiplying both sides in the first equality of (3.4) by $I + \frac{k}{2}A = R_{\frac{k}{2}}^{-1}$, using that $R_{\frac{k}{2}}X^{j-1} = (I - \frac{k}{2}A(I - \frac{k}{2}AR_{\frac{k}{2}}))X^{j-1}$ and then rearranging. We leave the details to the reader.

Subtracting (5.26) from (1.11), taking inner products with $X_{\text{be}}^j - X^j$ and rearranging yields

$$(5.27) \quad \begin{aligned} & \frac{1}{2}(\|X_{\text{be}}^j - X^j\|^2 - \|X_{\text{be}}^{j-1} - X^{j-1}\|^2 + \|X_{\text{be}}^j - X^j - X_{\text{be}}^{j-1} + X^{j-1}\|^2) \\ & + k\|X_{\text{be}}^j - X^j\|_1^2 \\ & = \frac{k}{2}\langle X^j - X^{j-1}, X_{\text{be}}^j - X^j \rangle_1 + \frac{k^2}{4}\langle AR_{\frac{k}{2}}X^{j-1}, X_{\text{be}}^j - X^j \rangle_1 \\ & \quad - k\langle f(X_{\text{be}}^j) - f(X^j), X_{\text{be}}^j - X^j \rangle \\ & \quad - k\langle A^{-1/2}(f(X^j) - f_k(R_{\frac{k}{2}}X^{j-1})), A^{1/2}(X_{\text{be}}^j - X^j) \rangle \\ & =: k(e_1^b + e_2^b + e_3^b + e_4^b). \end{aligned}$$

It is easily seen that

$$(5.28) \quad |e_1^b| \leq C\|X^j - X^{j-1}\|_1^2 + \epsilon\|X_{\text{be}}^j - X^j\|_1^2,$$

$$(5.29) \quad |e_2^b| \leq Ck^4\|AR_{\frac{k}{2}}X^{j-1}\|_1^2 + \epsilon\|X_{\text{be}}^j - X^j\|_1^2,$$

$$(5.30) \quad e_3^b \leq \beta^2\|X_{\text{be}}^j - X^j\|^2,$$

$$(5.31) \quad |e_4^b| \leq C\|A^{-1/2}(f(X^j) - f(J_k R_{\frac{k}{2}}X^{j-1}))\|^2 + \epsilon\|X_{\text{be}}^j - X^j\|_1^2.$$

By (1.7), the linear growth bound (3.24) of J_k , and the contractivity of $R_{\frac{k}{2}}$ in \dot{H}^1 , we have

$$(5.32) \quad \begin{aligned} & \|A^{-1/2}(f(X^j) - f(J_k R_{\frac{k}{2}} X^{j-1}))\| \\ & \leq C \left(\|X^j\|_1^2 + \|X^{j-1}\|_1^2 + 1 \right) \|X^j - J_k R_{\frac{k}{2}} X^{j-1}\|. \end{aligned}$$

From the triangle inequality, (3.34), (3.14) and (1.8) it follows that

$$(5.33) \quad \begin{aligned} \|X^j - J_k R_{\frac{k}{2}} X^{j-1}\| & \leq \|X^j - X^{j-1}\| + k \|AR_{\frac{k}{2}} X^{j-1}\| + k \|f_k(R_{\frac{k}{2}} X^{j-1})\| \\ & \leq \|X^j - X^{j-1}\| + k^{1/2} \|X^{j-1}\|_1 + k(\|X^{j-1}\|_1^3 + 1). \end{aligned}$$

Thus, from (5.31)–(5.33) we get

$$(5.34) \quad \begin{aligned} |e_4^b| & \leq C \left(P_4(\|X^j\|_1, \|X^{j-1}\|_1) \|X^j - X^{j-1}\|^2 + k P_{10}(\|X^{j-1}\|, \|X^j\|) \right) \\ & \quad + \epsilon \|X_{\text{be}}^j - X^j\|_1^2 \end{aligned}$$

with P_4 and P_{10} being a positive polynomials of degree 4 and 10, respectively. Inserting the bounds in (5.28)–(5.30) and (5.34) into (5.27) we conclude that

$$\begin{aligned} & \frac{1}{2} \left(\|X_{\text{be}}^j - X^j\|^2 - \|X_{\text{be}}^{j-1} - X^{j-1}\|^2 \right) + k \|X_{\text{be}}^j - X^j\|_1^2 \\ & \leq k \left\{ 3\epsilon \|X_{\text{be}}^j - X^j\|_1^2 + \beta^2 \|X_{\text{be}}^j - X^j\|^2 + C(\|X^j - X^{j-1}\|_1^2 + k^3 \|AR_{\frac{k}{2}} X^{j-1}\|_1^2 \right. \\ & \quad \left. + P_4(\|X^j\|_1, \|X^{j-1}\|_1) \|X^j - X^{j-1}\|^2 + k P_{10}(\|X^{j-1}\|_1, \|X^j\|_1) \right\}. \end{aligned}$$

By (3.14) we have that $k \|AR_{\frac{k}{2}} X^{j-1}\|_1^2 \leq \|AX^{j-1}\|^2$. Choosing $\epsilon = \frac{1}{3}$, we get, after summing up from $j = 1$ to $j = N$ and taking first the supremum over the discrete times and then the expectation, that

$$(5.35) \quad \begin{aligned} \mathbf{E} \sup_{0 \leq n \leq N} \frac{1}{2} \|X_{\text{be}}^n - X^n\|^2 & \leq \mathbf{E} \sum_{j=1}^N k \beta^2 \|X_{\text{be}}^j - X^j\|^2 \\ & \quad + Ck \left\{ k \sum_{j=1}^N \mathbf{E}(\|AX^{j-1}\|^2 + P_{10}(\|X^{j-1}\|, \|X^j\|)) \right\} \\ & \quad + C \sum_{j=1}^N k \mathbf{E} \|X^j - X^{j-1}\|_1^2 + C \mathbf{E} \sum_{j=1}^N k P_4(\|X^j\|_1, \|X^{j-1}\|_1) \|X^j - X^{j-1}\|^2. \end{aligned}$$

From Lemma 4.4 it follows that there is a constant $C_1 > 0$ depending on k_0 , t_N , $\mathbf{E}\|X_0\|_1^{10}$, $\mathbf{E}\|X_0\|_2^2$, and $\|A^{1/2}Q^{1/2}\|_{\text{HS}}$ that bounds the two sums in the parenthesis in the right hand side of (5.35). The next term is bounded in Lemma 5.7. For the last sum we use Hölder's inequality with exponents $5/2$ and $5/3$ to conclude that

$$(5.36) \quad \begin{aligned} & k \mathbf{E} \sum_{j=1}^N P_4(\|X^j\|_1, \|X^{j-1}\|_1) \|X^j - X^{j-1}\|^2 \\ & \leq \left(\mathbf{E} \sum_{j=1}^N k P_{10}(\|X^j\|_1, \|X^{j-1}\|_1) \right)^{2/5} \left(\mathbf{E} \sum_{j=1}^N k \|X^j - X^{j-1}\|^{10/3} \right)^{3/5}. \end{aligned}$$

By Lemmata 4.4 and 5.6 there exists a C_2 depending only on k_0 , t_N , $\mathbf{E}\|X_0\|_1^{10}$, and $\|A^{1/2}Q^{1/2}\|_{\text{HS}}$, such that the expression on the right hand side in (5.36) is bounded

by C_2k . Thus, for some $C > 0$, we have that

$$(5.37) \quad \mathbf{E} \sup_{0 \leq n \leq N} \|X_{\text{be}}^n - X^n\|^2 \leq \sum_{j=1}^N 2k\beta^2 \mathbf{E} \sup_{0 \leq i \leq j} \|X_{\text{be}}^i - X^i\|^2 + Ck$$

and, since $2k\beta^2 < 1$, the claim follows by Lemma 2.3. \square

Remark 5.9. *The proof of Theorem 5.8 gives a little bit more than what is stated in the theorem. Only $\mathbf{E}\|X_0\|_1^{10}$ needs to be bounded and not $\mathbf{E}\|X_0\|_1^{18}$.*

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